3^{rd} IFToMM International Symposium on Robotics and Mechatronics
2 – 4 October 2013, Singapore

Synthesis of Adjustable Planar and Spherical Four-link Mechanisms for Approximate Multi-path Generation

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This paper deals with the synthesis of planar and spherical adjustable four-link mechanisms whose coupler point can approximately trace two or more paths prescribed by a large number of points. A single adjustment in one of the mechanism parameters is used to approximately obtain the prescribed paths. A two stage approach is used to first determine the parameters of the driving dyad and then the parameters of the driven dyad. The use of novel and appropriate objective functions with minimum number of variables and a sequential quadratic programming (SQP) algorithm to search for the optimal mechanism parameters leads to quick and accurate synthesis of the four-link mechanisms.

INTRODUCTION

The problem of path generation by a point on the floating or coupler link of a four-link mechanism is very well studied (see, for example, Erdman and Sandor[1]). In a *point-to-point* path generation problem, for planar and spherical four-link mechanisms, the mechanism parameters can be obtained such that the chosen floating link point is made to pass through at most nine prescribed (also called precision) points.[2,3] For *continuous* path generation or when more than nine arbitrary points are prescribed, the point on the floating link point can be made to only *approximately* pass through the prescribed points, and in such a case the mechanism parameters can be obtained through an optimization problem where an appropriately defined objective function is minimised.[4,5] Adjustable mechanisms are capable of generating multiple paths with change in one or more of the mechanisms parameter and their advantages were recognized early (see, for example, textbook by Tao[6]). In planar four-link mechanisms, as adjustments are included, the number of design parameters become more than 10. Synthesis of adjustable planar four-link mechanism with prescribed points has been done by several researchers.[7–9] For continuous paths, genetic algorithm based optimisation has been used by Zhou and co-workers[10–12] and a two-stage synthesis using sequential quadratic programming (SQP) is reported in Peng and Sodhi.[13]

In a spherical mechanism all the links move on the surface of concentric spheres and in comparison to prescribed points in a plane for planar mechanism, points on the surface of a sphere are prescribed[a]. Compared to planar mechanisms, literature on synthesis of spherical mechanisms for finite number of precision points is limited (see references[14,15]). For continuous paths, method of constrained least squares optimization[16] for synthesis has been reported. Synthesis of adjustable spherical four-link mechanisms have also been discussed by some authors.[9,17–19]

The key step in any optimization based method is the formulation of an appropriate objective function. In this paper, least-squares circle-fitting (for planar) and a least-squares plane fitting (for spherical) objective function is suggested for optimal synthesis of adjustable four-link mechanisms. A single adjustment in the driven link is used to obtain different desired coupler paths and the parameters of the adjustable mechanism is obtained by using the SQP algorithm. The proposed formulation, as compared with existing approaches, is superior in terms of lesser number of optimization variables used and reduction of the search space. The proposed approach is illustrated with the synthesis of one planar and one spherical adjustable mechanism.

The paper is organized as follows: In section 2, we briefly describe the geometry of a typical planar and spherical four-link mechanism and present the well-known kinematic equations. In section 3, we present the formulations of the optimization scheme and in section 4, we present two examples to illustrate our approach. In section 5, we present the conclusions.

PLANAR AND SPHERICAL FOUR-LINK MECHANISMS

The planar and spherical four-link mechanisms **ABCDP** with its parameters is shown in figure 1. The planar four-link mechanism is very well known and its description can be found in any textbook on mechanisms. In a spherical mechanism, the axes of the four revolute joints intersect at the centre of the sphere, the point O, and the links are the arcs of great circles of the sphere. The spherical link length is the arc-length measured on the great circle between two ends of the link and for a sphere of unit radius, the link length is same as the central angle subtended at O by the arc on the great circle. All angles are dihedral angles, i.e., the angles are measured between two great circle planes and the line of intersection of the two circular planes is the axis about which the angle is measured. The coupler angle β is measured relative to BC. The parameters l_2, l_5, α_2 and α_5 are referred to as the *driving* side parameters, and the parameters β, l_3, l_4, α_3, α_4 and D are referred

[a]The points on a sphere can also be thought of as orientations of the floating or coupler link.

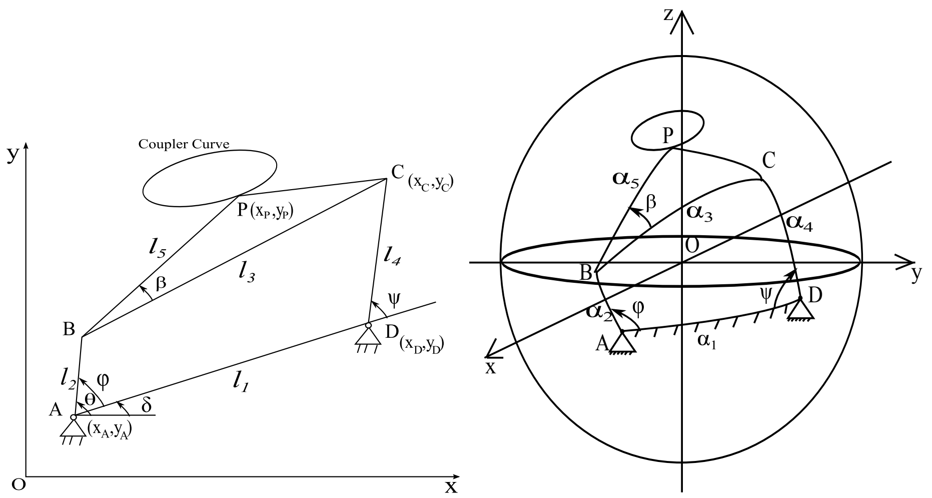

Fig. 1: Planar and speherical four-link mechanism.

to as the *driven* side parameters throughout the paper. The location of the driving crank pivot A remains unchanged in our approach. Figure 1 also shows the sketch of a typical coupler curve traced by a point P. We assume that a large number, 50 to 100, points are prescribed for each desired coupler path and we use a superscript on the mechanism parameters to indicate the particular path.

For any dyad (or a planar 2R manipulator) shown in figure 2, the workspace of the end-point lies in between two concentric circles.[20] Drawing parallels from the planar case, the workspace of the end-point of a spherical dyad lies between two coaxial spherical small circles, i.e., spatial circles about the same axis and different from the great circle on the surface of the sphere. The point A for planar (spherical)[b] is the optimized crank pivot location and A lies inside the desired coupler path if $l_2 > l_5\,(\alpha_2 > \alpha_5)$ and outside the path if $l_2 < l_5\,(\alpha_2 < \alpha_5)$.[12,13] This fact helps in choosing fixed pivot A. For N given points, $P_i, i = 1, 2, ..., N$, on each coupler curve, we define

$$
\begin{aligned}
&l_{max} = \max\{l_{P_1}, l_{P_2}, \ldots, l_{P_N}\} \quad \text{and} \quad l_{min} = \min\{l_{P_1}, l_{P_2}, \ldots, l_{P_N}\} \\
&\alpha_{max} = \max\{\alpha_{P_1}, \alpha_{P_2}, \ldots, \alpha_{P_N}\} \quad \text{and} \quad \alpha_{min} = \min\{\alpha_{P_1}, \alpha_{P_2}, \ldots, \alpha_{P_N}\} \\
&\text{where}, l_{P_i} = \overrightarrow{\boldsymbol{OP}} = ||\boldsymbol{r_{P_i}} - \boldsymbol{r_A}||, \alpha_{P_i} = \cos^{-1}(\boldsymbol{r_{P_i}} \cdot \boldsymbol{r_A}), for \quad i = 1, 2, \ldots, N
\end{aligned}
\tag{1}
$$

Since the coupler point P is on the driving dyad, the curve traced by P must lie in the area between two concentric circles with radii $l_2 + l_5$ $(\alpha_2 + \alpha_5)$ and $|l_5 - l_2|$ $(|\alpha_5 - \alpha_2|)$ centered at the crank pivot A. The lengths l_2 and l_5 are computed such that the above two concentric circles are tangential to the coupler curve. Hence, we

[b]The quantities in the bracket represent the corresponding quantity in the spherical domain.

must have,

$$\begin{aligned} l_{max} &= l_2 + l_5 \quad \text{and} \quad l_{min} = |l_5 - l_2| \\ \alpha_{max} &= \alpha_2 + \alpha_5 \quad \text{and} \quad \alpha_{min} = |\alpha_5 - \alpha_2| \end{aligned} \tag{2}$$

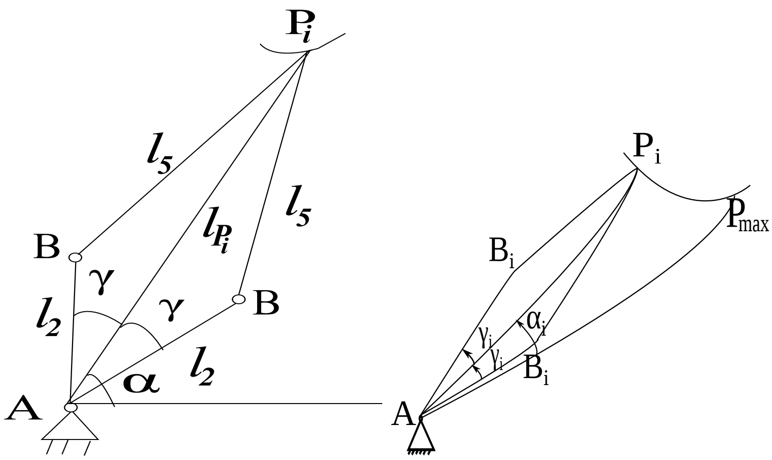

Fig. 2: Crank angle and a dyad in a plane and on the sphere.

In the planar case, for each P_i on the coupler path, the angle θ_i with respect to the $X-$ axis (see figure 1) is given as

$$\theta_i = \alpha_i \pm \gamma_i \tag{3}$$

where α_i is the orientation of $\overrightarrow{AP_i}$ in XY-plane relative to the positive global X-axis. The quantity γ_i is given by

$$\gamma_i = \cos^{-1}\left(\frac{l_2^2 + l_{P_i}^2 - l_5^2}{2 l_2 l_{P_i}}\right) \tag{4}$$

where $l_{P_i} = ||\overrightarrow{AP_i}||$, $0 \le \theta_i, \alpha_i \le 2\pi$ and $0 \le \gamma_i \le \pi$.

For the spherical case, the crank angle θ_i is also given by equation (3). It is measured with respect to the great circular plane containing α_{max} and P_{max} is the point on the spherical surface farthest from pivot A. We also have

$$\begin{aligned} &\boldsymbol{n_{max}} = \frac{\boldsymbol{r_A} \times \boldsymbol{r_{P_{max}}}}{||\boldsymbol{r_A} \times \boldsymbol{r_{P_{max}}}||} \quad \text{and} \quad \boldsymbol{n_{P_i}} = \frac{\boldsymbol{r_A} \times \boldsymbol{r_{P_i}}}{||\boldsymbol{r_A} \times \boldsymbol{r_{P_i}}||} \\ &\alpha_i = \cos^{-1}(\boldsymbol{n_{max}} \cdot \boldsymbol{n_{P_i}}) \quad \text{and} \quad \gamma_i = \cos^{-1}\left(\frac{\cos\alpha_5 - \cos\alpha_2 \cos\alpha_{P_i}}{\sin\alpha_2 \sin\alpha_{P_i}}\right) \\ &0 \le \alpha_i, \theta_i \le 2\pi \quad \text{and} \quad 0 \le \gamma_i \le \pi \end{aligned}$$

It may be noted that in one crank rotation, the crank crosses each of $l_{max}\,(\alpha_{max})$ and $l_{min}\,(\alpha_{min})$ lines once, thus dividing the rotation into two parts. The sign of γ_i in one $l_{max}\,(\alpha_{max})$ to $l_{min}\,(\alpha_{min})$ part is opposite to that in the remaining $l_{min}\,(\alpha_{min})$ to $l_{max}\,(\alpha_{max})$ part. Thus for each phase we have two sets of θ_i. If direction of rotation is not specified, appropriate θ_i must be chosen.

For each path point P_i, using the crank angle θ_i, B_i and C_i are as follows,

$$\begin{aligned} x_{B_i} &= x_A + l_2\cos(\theta_i) \quad \text{and} \quad y_{B_i} = y_A + l_2\sin(\theta_i) \\ x_{C_i} &= x_{B_i} + \frac{l_3}{l_5}\left[(x_{P_i} - x_{B_i})\cos(\beta) + (y_{P_i} - y_{B_i})\sin(\beta)\right] \\ y_{C_i} &= y_{B_i} + \frac{l_3}{l_5}\left[(y_{P_i} - y_{B_i})\cos(\beta) - (x_{P_i} - x_{B_i})\sin(\beta)\right] \end{aligned} \tag{5}$$

For the spherical case,

$$\begin{aligned} &\boldsymbol{r_{B_{max}}} = \left[T_{\alpha_2}^{\boldsymbol{n_{max}}}\right]\boldsymbol{r_A}, \quad \boldsymbol{r_{B_i}} = \left[T_{\theta_i}^{\boldsymbol{r_A}}\right]\boldsymbol{r_{B_{max}}} \quad \text{and} \quad \boldsymbol{n_{C_i}} = \frac{\boldsymbol{r_{B_i}} \times \boldsymbol{r_{P_i}}}{||\boldsymbol{r_{B_i}} \times \boldsymbol{r_{P_i}}||} \\ &\boldsymbol{r_{C_i}} = \left[T_{-\beta}^{\boldsymbol{n_{B_i}}}\right]\left[T_{\alpha_3}^{\boldsymbol{n_{C_i}}}\right]\boldsymbol{r_{B_i}} \end{aligned} \tag{6}$$

where, $\boldsymbol{r_p} = \overrightarrow{\boldsymbol{OP}}$ is the position vector of point P, $[T_\delta^{\boldsymbol{n}}]$ is the rotation matrix,[20] $\boldsymbol{n}$ is an unit vector corresponding to the axis of rotation and δ is the angle of rotation about $\boldsymbol{n}$ in counter-clockwise direction. In the next section, we state the various objective functions used for optimization.

FORMULATION OF OBJECTIVE FUNCTIONS

The main idea used in the optimization is that the locus traced by the point C, as the point P moves along the coupler path, is a circular arc for both planar and spherical cases. The objective function is the residual error obtained by *circle fitting* all the points C_i corresponding to coupler path points P_i and this is minimized in the planar case. The algorithm used for least-squares circle fitting is from Gander et al.[21] As the intersection of a plane and sphere results in a circle, the points C_i are plane fitted in the spherical case. The least squares fitting algorithm used in plane fitting is similar to circle fitting algorithm given in.[21] The algebraic fitting or linear problem in Gander et al.[21] is replaced by $f = ax + by + cz + d$ where (a, b, c, d) which define the plane. The geometric fitting or non-linear least squares problem is replaced by f_s given below as the objective functions. The design procedure is divided into two stages, **Stage I** deals with design of the driving dyad and **Stage II** deals with the computations of the remaining design parameters. It may noted that the two stage process does not restrict the solution space – the Stage I is a necessary condition for the driving dyad. The two stage approach has also been used by Peng and Sodhi.[13]

Stage I: Synthesis of driving dyad

To first design the driving dyad ABP, the optimal driving crank pivot location A needs to be determined. As the workspace of the driving dyad for both planar and spherical remains fixed for all the given paths, l_{max} and l_{min}, remain fixed throughout. The optimization problem can be formulated as follows

$$Minimize : S\left(x_A, y_A\right), \quad S_s\left(\kappa, \tau\right) = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \left[\left(l^i_{max} - l^j_{max}\right)^2 + \left(l^i_{min} - l^j_{min}\right)^2 \right]$$
$$where \quad l^i_k = ||\boldsymbol{r_A} - \boldsymbol{r^i_{P_k}}||, l^i_{max} = \max\left\{l^i_1, l^i_2, \ldots, l^i_N\right\}, l^i_{min} = \min\left\{l^i_1, l^i_2, \ldots, l^i_N\right\} \tag{7}$$

Subject to the following constraints:
Constraint 1: Search space restriction

$$x_A \in [x_{min}, x_{max}] \quad \text{and} \quad y_A \in [y_{min}, y_{max}] \quad (\kappa \in [0, \pi] \quad \text{and} \quad \tau \in [0, 2\pi]) \tag{8}$$

Constraint 2: The crank angle should always increase or decrease as P advances along the coupler curve. The conditions for the counter-clockwise and clockwise rotation of the crank respectively are,

$$\theta^i_{(q+1)} - \theta^i_q > 0 \quad \text{or} \quad \theta^{\mathrm{i}}_{(\mathrm{q}+1)} - \theta^{\mathrm{i}}_{\mathrm{q}} < 0 \quad \text{for} \quad \mathrm{q} = 1, 2, \ldots, \mathrm{N}-1 \tag{9}$$

where, N is the total number of points P^i_q on the given i^{th} coupler path and each θ_q is calculated using theory given above. The quantities (κ, τ) are the spherical polar coordinates of A for the spherical case. It maybe noted that one of the conditions in equation (9) also needs to be satisfied.

The optimization is carried out using the SQP algorithm which converges to the local minimum nearest to the starting point. Hence, to get the best solutions the search space is divided into several sub-intervals and the mid-point of the each sub-interval is taken as the starting point for the optimization in the corresponding sub-interval. The method gives a single solution for each sub-interval and hence we get as many solutions as the number of sub-intervals. To sort out the best driving dyads we select solutions which have objective function value, S, less than a user chosen maximum value of error S_{max}. The best solutions for the driving dyad are used to synthesize the remaining part of the four-bar mechanism. It should be noted that **Stage I** optimization only gives the possible locations of fixed pivot A. The exact location of A and the remaining mechanism parameters are determined after performing **Stage II** optimization.

Stage II: Synthesis of driven dyad

Once the optimal location of fixed pivot is obtained, $l_2(\alpha_2)$ and $l_5(\alpha_5)$ are found out as per theory given above. For the optimization functions formulated below, f and f_s represents the planar and spherical cases respectively, (a, b, r) represent the optimal circle containing the rocker path in the planar case and (a, b, c, d) represent the optimal plane containing the rocker path in the spherical case.

Type I: Adjustable driven side link pivot

In this type for both planar and spherical the position of fixed pivot D is variable, but the length of link CD remains same. Since m paths are to be traced, the movable pivot C will trace m different circles with m different centres D_i but with the same radius. In the spherical case, all circles will be in planes equidistant from the centre

O. The optimization problem can be stated as below

$$
\begin{aligned}
Minimize: f\left(l_3, \beta\right) &= \sum_{i=1}^{m}\sum_{j=1}^{N}\left(\sqrt{(a_i - x^i_{C_j})^2 + (b_i - y^i_{C_j})^2} - r\right)^2 \\
f_s\left(\alpha_3, \beta\right) &= \sum_{i=1}^{m}\sum_{j=1}^{N}\left(\frac{a_i x_{C^i_j} + b_i y_{C^i_j} + c_i z_{C^i_j}}{\sqrt{a_i^2 + b_i^2 + c_i^2}} + d\right)^2
\end{aligned}
\tag{10}
$$

Type II: Adjustable driven side link length

In this the length of link CD is variable, but fixed pivot D remains unchanged for all the m paths. Hence, the movable pivot C will trace m circular arcs with different radii but with the same center D. In the spherical case, all circles will be in parallel planes. The optimization problem can be stated as below

$$
\begin{aligned}
Minimize: f\left(l_3, \beta\right) &= \sum_{i=1}^{m}\sum_{j=1}^{N}\left(\sqrt{(a - x^i_{C_j})^2 + (b - y^i_{C_j})^2} - r_i\right)^2 \\
f_s\left(\alpha_3, \beta\right) &= \sum_{i=1}^{m}\sum_{j=1}^{N}\left(\frac{a x_{C^i_j} + b y_{C^i_j} + c z_{C^i_j}}{\sqrt{a^2 + b^2 + c^2}} + d_i\right)^2
\end{aligned}
\tag{11}
$$

Subject to the following constraints:

$$\textit{Constraint 1}: \quad l_2 < l_3 \leq l_{3max} \quad (\alpha_2 < \alpha_3 \leq \alpha_{3max}) \tag{12}$$

$$\textit{Constraint 2}: \quad -\pi \leq \beta < \pi \tag{13}$$

$$\textit{Constraint 3}: \quad l_2 < l_4^i \quad \text{and} \quad l_2 < l_1^i \quad \left(\alpha_2 < \alpha_1^i \quad \text{and} \quad \alpha_2 < \alpha_4^i\right) \tag{14}$$

Constraint 4: For link CD to be a rocker, the angular sweep of link CD should be less than π radians or

$$\psi_{max} - \psi_{min} < \pi \tag{15}$$

Constraint 5: Grashof's criterion for crank-rocker type mechanism should be satisfied for each i^{th} path.

The cost function $f(f_s)$ is the least-square residue error obtained during circle (plane) fitting. To get the minimum $f(f_s)$ in the sub-interval, we need to get circle (plane) parameters at the optimum point in the sub-interval. During each iteration in the sub-interval, the optimization problem is converted into a non-linear least-squares problem with circle (plane) parameters as unknowns. The non-linear least-squares problem is solved using Gauss-Newton method which needs a starting value for the unknowns. The procedure for obtaining the unknowns is as follows:

Step 1: The set of C_i's for each coupler paths are separately circle (plane) fitted to obtain the circle (plane) parameters for each path.

Step 2: Non-linear least-squares problem given in (10) is formed with circle (plane) parameters as unknown variables. The starting value of the unknowns are the output of the Step 1. The starting value for the common parameter is the average of the values obtained for it in Step 1. Step 2 gives the values for the unknowns for the corresponding optimization iteration in the sub-interval. Using these values and equations (10) and (11), we can calculate cost function $f(f_s)$ for each optimization iteration. The remaining mechanism parameters can be calculated using trigonometry and vector algebra after **Stage 2**. The procedure given above can also be applied for$l_2 > l_5\,(\alpha_2 > \alpha_5)$.

NUMERICAL RESULTS

In this section, we present two examples, one each from planar and spherical, to illustrate multi-path generation by adjustable four-link mechanisms. The optimization was done using *fmincon* function of MATLAB.[22]

Example 1:

This example has been originally studied in Peng and Sodhi.[13] We show that our approach using less number of search variables yields similar results. The given 20 data points are refined to 50 points using spline interpolation. We have $x_A \in [-20, 20]$ and $y_A \in [-20, 20]$ and both intervals are divided into 20 sub-intervals each. The intervals $l_3 \in [3, 20]$ and $\beta \in [-\pi, \pi]$ are divided into 4 sub-intervals each. The value of S_{smax} is chosen to be 0.003 which results in 9 possible locations of pivot A. The optimization results using the adjustable pivot method are as follows: $A\,(0.000, -18.551)$, $l_1^1 = 9.830$, $l_1^2 = 15.254$, $\delta_1 = 2.577$rad, $\delta_2 = 1.807$rad, $l_2 = 3.0562$, $l_3 = 9.874$, $l_4 = 9.900$, $\beta = -0.022$rad, $l_5 = 19.400$, $S = 0.000733$, $f = 0.182$. We have compared the total path error (E_{Total}= error in path 1 + error in path 2) for the example in.[13] The numbers are, Our approach, $E_{Total} = 3.800$, Reference[13] approach, $E_{Total} = 3.500$. The numbers are similar.

The best fitting circles for C and the synthesised paths are shown in figure 3.

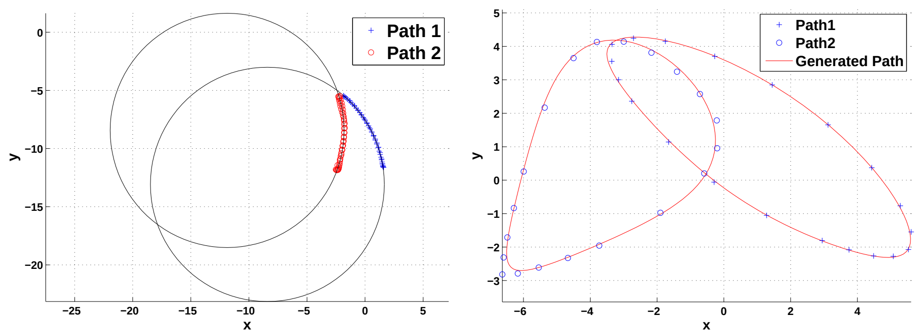

Fig. 3: Numerical results for planar example.

Example 2:

For this example, we transform the points in the Example 1 to the surface of a sphere centered at $(-0.92, 3.46, 0)$ and with radius 11. The points are then normalized to be on the unit sphere. The search space, $\kappa \in [0, \pi]$ is divided into 5 intervals and

$\tau \in [0, 2\pi]$ is divided into 10 intervals. The quantity $\alpha_3 \in [0.3\text{rad}, 2.5\text{rad}]$ is divided into 5 intervals and $\beta \in [-\pi, \pi]$ is divided in 3 intervals. The quantity S_{max} is chosen to be 10^{-4} which results in 23 dyads to be chosen. The optimization results are as follows:
$\kappa = 1.349$rad, $\tau = 4.755$rad, Coordinates of $A = (0.0417, -0.9746, 0.2201)$, $\alpha_2 = 0.3044$rad, $\alpha_5 = 1.1256$rad, $\alpha_1^1 = 0.5146$rad, $\alpha_1^2 = 0.9520$rad, $\alpha_3 = 0.6283$rad, $\alpha_4 = 0.6290$rad, $\beta = 0.0331$rad, Coordinates of $D_1 = (-0.3503, -0.7971, 0.4919)$ and $D_2 = (-0.6662, -0.4983, 0.5549)$, $S_s = 10^{-7}$, $f_s = 0.0083$.
The best fitting circles for C and the synthesized paths are shown in figure 4.

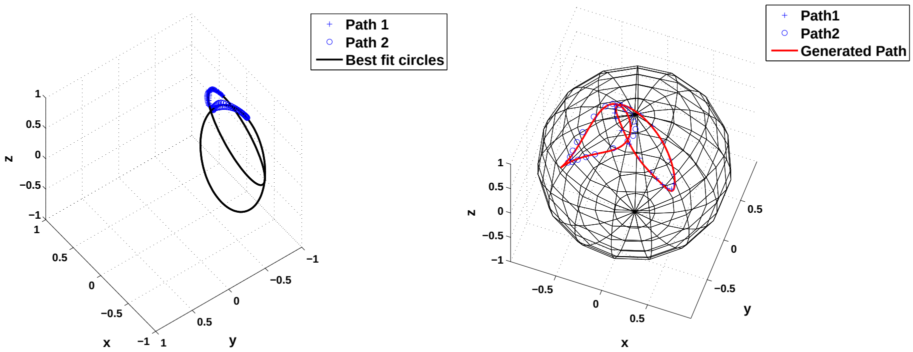

Fig. 4: Numerical results for spherical example.

CONCLUSIONS

This paper dealt with optimization based schemes for synthesis of planar and spherical four-link mechanisms. A sequential quadratic programming approach is used to synthesise planar and spherical adjustable four-link mechanisms for multi-path generation. The method presented in this paper uses less number of variables as compared to existing approaches. In this paper only driven side adjustment is presented. However, the approach of this paper can be easily extended for adjustments in the driving, coupler and remaining driven side parameter, and these have not been presented here due to space restrictions.

This work is continuing and we are in the process of building a prototype for Examples 1 and 2 to validate the theory presented in this work.

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