A differential-geometric analysis of singularities of point trajectories of serial and parallel manipulators

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Abstract

In this paper, we present a differential-geometric approach to analyze the singularities of task space point trajectories of two and three-degree-of-freedom serial and parallel manipulators. At non-singular configurations, the first-order, local properties are characterized by metric coefficients, and, geometrically, by the shape and size of a velocity ellipsoid or an ellipsoid. At singular configurations, the determinant of the matrix of metric coefficients is zero and the velocity ellipsoid degenerates to an ellipse, a line or a point, and the area or the volume of the velocity ellipse or ellipsoid becomes zero. The degeneracies of the velocity ellipsoid or ellipse gives a simple geometric picture of the possible task space velocities at a singular configuration. To study the second-order properties at a singularity, we use the derivatives of the metric coefficients and the rate of change of area or volume. The derivatives are shown to be related to the possible task space accelerations at a singular configuration.

In the case of a parallel manipulators, singularities may lead to either loss or gain of one or more degrees-of-freedom. For loss of one or more degrees-of-freedom, the possible velocities and accelerations are again obtained from a modified metric and derivatives of the metric coefficients. In the case of a gain of one or more degrees-of-freedom, the possible task space velocities can be pictured as growth to lines, ellipses, and ellipsoids. The theoretical results are illustrated with the help of a general spatial 2R manipulator and a three-degree-of-freedom RPSSPR-SPR parallel manipulator.

1 Introduction

Evaluation of singularities plays an important role in several aspects of robotics including design, trajectory planning, and control. Much of the past research in the area of singularities of manipulators have been related to the study of manipulator configurations resulting in singularities (see, for example, (Wang and Waldron 1987; Litvin, Zhang, Castelli and

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Innocenti 1990; Hunt 1986; Martinez, Alvarado and Duffy 1994)), enumeration and classification of kinematic structure of manipulators and mechanisms with singular configurations(see, for example, (Lipkin and Pohl 1991; Karger 1995; Karger 1996*a*; Sugimoto, Duffy and Hunt 1982; Gosselin and Angeles 1990; Litvin, Fanghella, Tan and Zhang 1986; Merlet 1991)), novel designs of manipulators and wrists, including use of redundancy, that would exclude singularities from the useful portion of the workspace (see, for example, (Stanisic and Duta 1990; Tchnon and Matuszok 1995; Shamir 1990)), analysis of singular sets for serial manipulators(see, for example, (Karger 1996*b*)) and planning of trajectories at singularities(see, for example, (Chevallereau 1998; Lloyd 1996; Nenchev, Tsumaki, Uchiyama, Senft and Hirzinger 1996; Nenchev and Uchiyama 1996; Martinez et al. 1994; Sardis, Ravani and Bodduluri 1992)). A serial manipulator is said to be in a singular configuration when the manipulator Jacobian matrix looses rank. Most of the approaches towards singularity analysis, in the references mentioned, involve use of linear algebra based techniques such as singular value decomposition and use of Taylor series expansion of certain measures of manipulability around a singular configuration. A few researchers have also used concepts from singularities of maps(Golubitsky and Guillemin 1973) to characterize singularities of point trajectories and their bifurcations (see (Kieffer 1992; Kieffer 1994)) and singularities of non-redundant kinematics(Tchnon and Muszynski 1997).

In this paper, we present a differential-geometric approach towards singularity analysis of non-redundant point trajectories traced by serial and parallel manipulators. The paper differs from the works mentioned above in that a) it develops a differential-geometric method for local characterization of singularities, and b) the method is applied to both serial and parallel manipulators. The main idea of this paper is based on the well-known concept of a *metric* on a manifold and the associated concepts of a velocity ellipsoid or ellipse(Ghosal and Roth 1987), whose size and shape characterizes the local, first-order properties of non-singular point trajectories. At singular positions, the definition of the metric is no longer valid and the velocity ellipsoid degenerates to an ellipse, line or a point. These degenerate forms of the ellipsoids or ellipses give clear geometric picture of the possible task space velocities at a singular configuration, and we present simple algorithms involving eigenvalues and eigenvectors of the matrix of metric coefficients to obtain the degenerate forms of the ellipsoids or ellipses. For second and higher-order properties, we consider the derivatives of the metric coefficients and the rate of change of the volume of the ellipsoid since familiar concepts such as curvature is not defined at a singularity. The rate of change of the metric coefficients are related to the possible task space accelerations at a singular configuration.

In the case of parallel manipulators, singularities can lead to either *loss* or *gain* of one or more degrees-of-freedom. We use the concept of velocity ellipsoids and ellipses, for loss of degree-of-freedom in parallel manipulators, by deriving a suitable metric. For gain of one or more degrees-of-freedom, we show that the singularity can be pictured as a "growth" to lines, ellipses and ellipsoids. The results of this paper, in addition, to their theoretical interests in kinematics of manipulators, have applications in trajectory planning and control.

The paper is organized as follows: In section 2, we briefly present the concept of a metric, the associated velocity ellipse and ellipsoid, the derivatives of the metric coefficients and then discuss its usefulness for differential analysis of point trajectories traced out by non-redundant, serial and parallel manipulators. In section 3, we discuss singularities of point trajectories traced out by two and three-degree-of-freedom serial and parallel manipulators by considering the metric coefficients, their derivatives and the rate of change of the volume of the velocity ellipsoid. In section 4, we illustrate our theory with the help of a general spatial 2R and a three-degree-of-freedom RPSSPR-SPR parallel manipulator. Finally, in section 5, we present the conclusions.

2 Mathematical Formulation

The trajectory traced by a point in a moving rigid body can be expressed as a set of equations giving the coordinates of the point in the terms of the n independent motion parameters. Assuming that the coordinates of the point are the Cartesian coordinates, (x, y, z), and the n independent motion parameters are denoted by $\theta_i,\ i = 1, 2, ..., n$, the set of equations can be written in a symbolic form as

$$(x, y, z)^T = \boldsymbol{\psi}(\theta_1, ..., \theta_n) \tag{1}$$

In the case of a manipulator, the vector function $\boldsymbol{\psi}$ depends on the point chosen on the end-effector, the geometry and structure of the manipulator and its dimensions. The function $\boldsymbol{\psi}$ and can be thought of as a mapping which takes points in the motion parameter space, $(\theta_1, ..., \theta_n)$, to points in the 3D (Euclidean) space of the motion. These equations are the familiar *direct kinematics* equations for a manipulator.

In the case of serial manipulators with n degrees of freedom, the n motion parameters are the rotations or translations at the joints and are independently actuated. In the case of parallel manipulators and closed-loop mechanisms, not all the n motion parameters are actuated and m of them may be passive. In such a case the degree of freedom of the parallel manipulator or the closed-loop mechanism is $(n-m)$, and in addition to the above equations, we have m independent constraint equations of the form

$$\boldsymbol{\eta}(\theta_1, ..., \theta_n) = \mathbf{0} \tag{2}$$

where $\boldsymbol{\eta}(.)$ denotes the m constraint functions $\eta_i(.), i = 1, 2, .., m$[1].

2.1 Differential kinematics of serial manipulators at non-singular points

In the case of serial manipulators, the velocity at any point, $\mathbf{p}$, on the point trajectory can be written as

$$\mathbf{v} = \sum_{i=1}^{n} \boldsymbol{\psi}_i \dot{\theta}_i \tag{3}$$

where $\dot{\theta}_i$ is the time derivative of θ_i and $\boldsymbol{\psi}_i$ is the first partial derivative of $\boldsymbol{\psi}$ with respect to θ_i or $\partial\boldsymbol{\psi}/\partial\theta_i$. The partial derivatives are evaluated at $\mathbf{p}$. The above equation can also be written in terms of the matrix of first partial derivatives or the Jacobian matrix as

$$\mathbf{v} = [J(\boldsymbol{\psi})]_{\mathbf{p}} \dot{\boldsymbol{\theta}} \tag{4}$$

where $\boldsymbol{\theta}$ is the vector $(\theta_1, ..., \theta_n)^T$ and $[J(\boldsymbol{\psi})]_{\mathbf{p}}$ is the Jacobian matrix of $\boldsymbol{\psi}$ evaluated at $\mathbf{p}$. By varying $\dot{\boldsymbol{\theta}}$, we can get any arbitrary velocity $\mathbf{v}$ at $\mathbf{p}$. It is more instructive to look at the variation of $\mathbf{v}$ with a normalizing constraint of the form $\dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} = k^2$. For $k = 1$, we have a *unit speed motion* and by varying k one can get all possible velocities, $\mathbf{v}$, at the point $\mathbf{p}$ under consideration[2].

The dot product of the velocity with itself can be written as

$$\mathbf{v} \cdot \mathbf{v} = \dot{\boldsymbol{\theta}}^T [g] \dot{\boldsymbol{\theta}} \tag{5}$$

where the matrix elements g_{ij} are the dot products $(\boldsymbol{\psi}_i \cdot \boldsymbol{\psi}_j), i, j = 1, 2, .., n$. The matrix $[g]$, equal to $[J(\boldsymbol{\psi})]^T [J(\boldsymbol{\psi})]$, is symmetric and positive definite and its elements(in the language of differential geometry) define the **metric** in the tangent space(Millman and Parker 1977). We make the following observations from the definition of $[g]$ and equation (5):

[1] In this paper, we restrict ourselves to non-redundant manipulators, i.e., $n \leq 3$ for serial manipulators and $(n-m) \leq 3$ for parallel manipulators and closed-loop mechanisms.

[2] For a prismatic joint, with joint variable denoted by d_i, we use $\frac{\dot{d}_i}{d_{i\,max}}$ where $d_{i\,max}$ is the maximum value d_i can take. This ensures that all terms are dimensionally the same.

- If $[g]$ is non-singular(i.e, $\det[g] \neq 0$), then we can write

$$\mathbf{v}^T([J][g]^{-1})([J][g]^{-1})^T\mathbf{v} = \dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} \tag{6}$$

  The matrix $([J][g]^{-1})([J][g]^{-1})^T$ is symmetric and positive definite, and for a constraint of the form $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = 1$, the tip of the velocity vector $\mathbf{v}$ lies on an ellipsoid. For two-degree-of-freedom motions the tip of the vector $\mathbf{v}$ lies on an ellipse in the tangent plane. In classical differential geometry of surfaces, this ellipse defines the so-called *Tissot's indicatrix*(see pages 145-155 of Strubecker(Strubecker 1969)).

- The maximum and minimum values of $\mathbf{v}^2$ subject to constraint $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = 1$ can be obtained by solving

$$\partial\mathbf{v}^{*2}/\partial\dot{\theta}_1 = \partial\mathbf{v}^{*2}/\partial\dot{\theta}_2 = 0 \tag{7}$$

  where $\mathbf{v}^{*2}$ is given as

$$\mathbf{v}^{*2} = \dot{\boldsymbol{\theta}}^T[g]\dot{\boldsymbol{\theta}} - \lambda(\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} - 1) \tag{8}$$

  The above reduces to solving the eigenvalue problem

$$[g]\dot{\boldsymbol{\theta}} - \lambda\dot{\boldsymbol{\theta}} = 0 \tag{9}$$

  The maximum and minimum $|\mathbf{v}|$ in terms of the maximum and minimum eigenvalues of $[g]$, λ_{max} and λ_{min}, are given as

$$\begin{aligned} |\mathbf{v}|_{max} &= \sqrt{\lambda_{max}} \\ |\mathbf{v}|_{min} &= \sqrt{\lambda_{min}} \end{aligned} \tag{10}$$

  The directions of the maximum and minimum velocities are related to the eigenvectors of $[g]$ and are along the vectors $[J(\boldsymbol{\psi})]\dot{\boldsymbol{\theta}}_i$, $i = 1, 2, 3$ where $\dot{\boldsymbol{\theta}}_i$ is the eigenvector corresponding to eigenvalue λ_i.

  The maximum, minimum, and intermediate values of $|\mathbf{v}|$ are along the three principal axes of the ellipsoid and determine the shape of the ellipsoid(for an ellipse there are only a maximum and a minimum). If the normalization $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = k^2$ is used then the maximum and minimum values are scaled by k but the shape of the velocity ellipsoid(or ellipse) doesn't change.

- The volume (area in case of ellipse) is proportional to $\sqrt{\det[g]}$. For an ellipse the area is $k\pi\sqrt{\det[g]}$ and for an ellipsoid, the volume is given by k $(4\pi/3)\sqrt{\det[g]}$. It may be noted that $\det[g]$ is equal to the product of the eigenvalues.

- Yoshikawa(Yoshikawa 1985) introduced an useful manipulability measure $\sqrt{\det([J][J]^T)}$ which has been used extensively by several researchers for resolution of redundancy(Nakamura 1991). Several authors have also used the singular values of $[J]$ to analyze the first order properties. However, $\det[g]$ used in this paper and $\det([J][J]^T)$ (and the square root of eigenvalues of $[g]$ and the singular values of $[J]$) have significant differences. We list some of them below.

  1) The elements of $[g]$ and $\det[g]$ define distance, angle and elemental area(or volume) on a manifold whereas the matrix $[J][J]^T$ comes from a least squares type of solutions to a system of linear equations[3]. In this paper, our approach is from a differential-geometric perspective and not from linear algebra, and the focus of the paper is on singularities where the definition of a metric on a manifold breaks down and $\det[g]$ equals zero.

  2) As shown later, the quantity $\det[g]$ naturally occurs when we consider second and higher-order properties of a manifold such as the Gaussian curvature. It is not clear how the the manipulability measure can be used to study second and higher-order properties of a manifold.

  3) The elements of $[g]$ and $\det[g]$ are well-defined for all non-redundant manipulators and mechanisms. The manipulability measure is more suited for redundant manipulators since $\det([J][J]^T)$ is zero for a non-redundant spatial 2R manipulator. It may be noted that $\det([J]^T[J])$ is always zero for a redundant manipulator.

We next discuss parallel manipulators and closed-loop mechanisms.

2.2 Differential kinematics of parallel manipulators at non-singular points

As mentioned before, in the case of a parallel manipulator or closed-loop mechanism not all the n joints are actuated and there are m constraint equations of the form (2). We denote the $(n-m)$ actuated joints by the vector $\boldsymbol{\theta}$ and the m passive joints by the vector $\boldsymbol{\phi}$. The velocity vector, $\mathbf{v}$, at any point $\mathbf{p}$ on the point trajectory traced by a parallel manipulator or a closed-loop mechanism can be written as

$$\mathbf{v} = [J]\dot{\boldsymbol{\theta}} + [J^*]\dot{\boldsymbol{\phi}} \tag{11}$$

[3]The solution to a set of linear equations(Golub and Loan 1989) $Ax = b$, $A \in \Re^{m \times n}$, $x \in \Re^n$, $b \in \Re^m$, is given as $x = (A^TA)^{-1}A^Tb$ when $m \geq n$ and $x = A^T(AA^T)^{-1}b$ when $m \leq n$. $(A^TA)^{-1}A^T$ and $A^T(AA^T)^{-1}$ are called the pseudo-inverse of A.

where the columns of $[J]$ are the partial derivatives of $\boldsymbol{\psi}$ with respect to the $n-m$ actuated joint variables θ_i and the columns of $[J^*]$ are the partial derivatives of $\boldsymbol{\psi}$ with respect to the m passive variables ϕ_i. The dimensions of $[J]$ and $[J^*]$ are $(dim(\mathbf{v}) \times (n-m))$ and $(dim(\mathbf{v}) \times m)$ respectively. By differentiating the m constraints equations (2), we get

$$\mathbf{0} = \sum_i^m \boldsymbol{\eta}_i \dot{\theta}_i \tag{12}$$

where $\boldsymbol{\eta}_i$ is the partial derivative $\partial\boldsymbol{\eta}/\partial\theta_i$. Again assuming that the first $(n-m)$ θ_i's are actuated and the rest are passive, we can rearrange these m equations in the form

$$\mathbf{0} = [K]\dot{\boldsymbol{\theta}} + [K^*]\dot{\boldsymbol{\phi}} \tag{13}$$

where the columns of $[K]$ are the first $(n-m)$ $\boldsymbol{\eta}_i$'s and the columns of $[K^*]$ are the last m $\boldsymbol{\eta}_i$'s. It may be noted that $[K^*]$ is always a square matrix of dimension $m \times m$.

Assuming that $\det[K^*] \neq 0$, we can solve for $\dot{\boldsymbol{\phi}}$ from equation (13) as

$$\dot{\boldsymbol{\phi}} = -[K^*]^{-1}[K]\dot{\boldsymbol{\theta}} \tag{14}$$

and on substituting in equation (11), we get

$$\mathbf{v} = ([J] - [J^*][K^*]^{-1}[K])\dot{\boldsymbol{\theta}} \tag{15}$$

where $\boldsymbol{\theta}$ is the vector of $n-m$ actuated joint variables.

Equation (15) is similar to equation (3) for serial manipulators and we can define the matrix of metric coefficients for a parallel manipulator, $[g^*]$, as

$$[g^*] = ([J] - [J^*][K^*]^{-1}[K])^T([J] - [J^*][K^*]^{-1}[K]) \tag{16}$$

The matrix $[g^*]$ is symmetric and positive definite[4] and we can again state that for a normalization constraint of the form $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = k^2$, the tip of the velocity vector lies on an ellipsoid(ellipse). The shape and size of the ellipsoid(ellipse) is again determined by the eigenvalues of the matrix $[g^*]$.

[4] $[g^*]$ is clearly symmetric since it is of the form $[A]^T[A]$. It is also positive definite provided that $\det[K^*] \neq 0$ and $([J] - [J^*][K^*]^{-1}[K])$ is non-singular.

2.3 Higher order properties at non-singular points

The velocity vector, $\mathbf{v}$, derived from the first derivative of the mapping function or the elements of the matrix $[g]$ (or $[g^*]$) determine the first order properties of the point trajectories. For the second-order properties we consider the acceleration vector given in terms of the first and second partial derivatives of $\boldsymbol{\psi}$ as

$$\mathbf{a} = \sum_{i=1}^{n} \boldsymbol{\psi}_i \ddot{\theta}_i + \sum_{i,j=1}^{n} \boldsymbol{\psi}_{ij} \dot{\theta}_i \dot{\theta}_j \tag{17}$$

When the number of independent θ_i is two and the point trajectory is in $\Re^3$, we get using well known results from differential geometry,

$$\mathbf{a} = \sum_{i=1}^{2} \boldsymbol{\psi}_i \ddot{\theta}_i + \sum_{i,j=1}^{2} (\Gamma_{ij}^k \boldsymbol{\psi}_k + L_{ij}\mathbf{n}) \dot{\theta}_i \dot{\theta}_j \tag{18}$$

where $\mathbf{n}$ is the normal vector, L_{ij}'s are the dot products $\boldsymbol{\psi}_{ij} \cdot \mathbf{n}, i, j = 1, 2$, and the six Γ_{ij}^k are called *Christoffel symbols*(Millman and Parker 1977). The Christoffel symbols are given as

$$\Gamma_{ij}^k = \sum_{l=1}^{2} (\boldsymbol{\psi}_{ij} \cdot \boldsymbol{\psi}_l) g^{lk} \quad i, j, k = 1, 2 \tag{19}$$

where g^{lk} is the (l, k) element of $[g]^{-1}$. It may be noted that the six *Christoffel symbols* can be expressed as partial derivatives of the metric coefficients g_{ij}'s(or g_{ij}^*'s for parallel manipulators)(Millman and Parker 1977).

The second order properties are completely determined by the elements g_{ij}, L_{ij}, and Γ_{ij}^k. The local geometry of the surface is determined by the *Gaussian curvature* given by

$$K = \frac{\det[L]}{\det[g]} \tag{20}$$

and K can also be derived only in terms of the metric coefficients, g_{ij} and their first partial derivatives(g_{ij}^* and its first partial derivatives for parallel manipulators)(Millman and Parker 1977).

In case the point trajectory is a solid region in $\Re^3 (n = 3)$, the tangent space is of the same dimension as the space of the motion and there is no notion of a normal vector. One can consider 2D sections (surfaces) of the solid region and compute the Gaussian curvature of each of the sections by computing the appropriate L_{ij}'s and the $\det[g]$'s. Another general approach is to compute the first and second partial derivatives of g_{ij}'s(g_{ij}^* in case of parallel manipulators) with respect to the motion parameters and define the Riemannian curvature tensor(Millman and Parker 1977) as

$$R_{ijkl} = (1/2)(\frac{\partial^2 g_{il}}{\partial\theta_j\partial\theta_k} + \frac{\partial^2 g_{jk}}{\partial\theta_i\partial\theta_l} - \frac{\partial^2 g_{ik}}{\partial\theta_j\partial\theta_l} - \frac{\partial^2 g_{jl}}{\partial\theta_i\partial\theta_k}) + g^{st}(\Gamma_{jsk}\Gamma_{itl} - \Gamma_{jsl}\Gamma_{itk}) \qquad (21)$$

where g^{st} is the (s,t) element of $[g]^{-1}$ and

$$\Gamma_{ijk} = \frac{1}{2}(\frac{\partial g_{jk}}{\partial\theta_i} + \frac{\partial g_{ik}}{\partial\theta_j} - \frac{\partial g_{ij}}{\partial\theta_k}) \qquad (22)$$

The above rank four tensor has the properties of curvature since one can show that if all the R_{ijkl} vanishes everywhere, then the volume of the velocity ellipsoid is constant everywhere similar to the case when the mapping $\boldsymbol{\psi}$ is linear and the surface is flat. The Gaussian curvature of a 2D subspace can also be computed as

$$K = \frac{R_{1212}}{\det[g]} \qquad (23)$$

It may be noted that in the expression of the Gaussian curvature (and Riemannian curvature) and the Christoffel symbols, $\det[g]$($\det[g^*]$ for parallel manipulators and closed-loop mechanisms) appears in the denominator. If $\det[g]$ or $\det[g^*]$ is zero, then we have a *singularity*. At a singularity, the Christoffel symbols and the Gaussian or the Riemannian curvatures are not defined[5], and hence, we cannot characterize the geometry of the point trajectory with Γ_{ij}^k, K or R_{ijkl}. In the next section, we analyze the behavior of $\det[g]$ and the derivatives of g_{ij}, to characterize the singularities in two and three-degree-of-freedom motions.

3 Singularity analysis

We start with singularity analysis for serial manipulators where, as discussed above, the singularity is related to the loss of rank of the matrix $[g]$. In the later part, we will discuss singularity analysis of parallel manipulators and closed-loop mechanisms, where we can not only have loss of rank of $[g^*]$ but also loss of rank of $[K^*]$.

3.1 Singularity analysis for serial manipulators

As mentioned in the previous section, the first-order properties at a non-singular point is characterized by a velocity ellipsoid which is in turn completely determined by the metric

[5]The curvatures and the Christoffel symbols are in the indeterminate form 0/0.

coefficients, g_{ij}. At a singularity $\det[g] = 0$ and the matrix $[g]$ looses rank. From the definition of the metric coefficients, the matrix $[g]$ is always square and symmetric, and hence the loss of rank of $[g]$ is characterized by one or more real eigenvalue of $[g]$ going to zero. It may be noted that the singular direction are the eigenvectors corresponding to the zero eigenvalues. We can have the following cases at a singularity as far as the task space velocity is concerned:

- One eigenvalue is zero, say $\lambda_1 = 0$ and λ_2,λ_3 non-zero. In this case, the ellipsoid degenerates to the inside of an ellipse, and the task space velocity along the direction corresponding to the zero eigenvalue will be zero.

- Two eigenvalues zero, say $\lambda_1 = \lambda_2 = 0$ and λ_3 non-zero. In such a case, the ellipsoid degenerates to a line, and the task space velocity in the plane spanned by the eigenvectors corresponding to the zero eigenvalues will be zero.

- All three eigenvalues zero. In such a case the ellipsoid degenerates to a point, and there can be no velocity in any direction.

To consider the possible accelerations, we first consider an arbitrary direction $\mathbf{s}$ in the tangent space. In general $\mathbf{s}$ can be written as $\sum_{i=1}^{n} c_i \boldsymbol{\psi}_i$ where c_i are constants not all zero. The acceleration along this arbitrary direction is given by

$$\mathbf{s} \cdot \mathbf{a} = \mathbf{c}^T [g] \ddot{\boldsymbol{\theta}} + \sum_{i=1}^{n} c_i \Gamma_{jki} \dot{\theta}_j \dot{\theta}_k \tag{24}$$

where $\mathbf{c}$ is the vector $(c_1, ..., c_n)^T$ and the Christoffel symbols, Γ_{jki} can be determined from the partial derivatives of the metric coefficients as in equation (22)[6]. In order to find the acceleration at the singular point, in the arbitrary direction, we have to evaluate both the terms. It may be noted that $[g]$ is of rank less than n at a singularity.

If $\mathbf{s}$ is a singular direction then $[g]\mathbf{c} = 0$, and from equation (24), the first term on the right-hand side is zero($[g]$ is symmetric). If the second term is non-zero, one can have non-zero acceleration along a singular direction with non-zero joint velocity. Furthermore the acceleration is a quadratic function of the joint velocities.

If the second term is zero, then one cannot have acceleration along the singular direction with finite joint velocities. In such a case, to study possible motions, we have to look at

[6]It may be noted that similar formula has been derived by Kager(Karger 1989)(see formula(19)) in the context of analyzing curvature properties of 6 degree-of-freedom robots.

higher-order analysis. The dot product of $\mathbf{s}$ with the jerk, $\dot{\mathbf{a}}$, can be written as

$$\mathbf{s}\cdot\dot{\mathbf{a}} = \mathbf{c}^T[g]\boldsymbol{\theta}''' + \sum_{i=1}^{n} c_i \Gamma_{jki}\dot{\theta}_j\ddot{\theta}_k + \sum_{i=1}^{n} c_i \boldsymbol{\psi}_i \cdot [\dot{\theta}_1^3 \boldsymbol{\psi}_{111} + ... + \dot{\theta}_n^3 \boldsymbol{\psi}_{nnn}] \tag{25}$$

where $(\ )'''$ denotes the third derivative with respect to t.

Along the singular direction, the first and second term are zero and one can have non-zero jerk with finite joint velocities if the third derivatives, $\boldsymbol{\psi}_{ijk}$ are non-zero. It may be noted that the jerk is a cubic function of the joint velocities.

We can also analyze singularities, by studying the behavior of $\det[g]$ near a singularity. We denote the values of $(\theta_1, .., \theta_n)^T$ which satisfy the singularity condition, $\det[g] = 0$, by $\boldsymbol{\theta}^*$, and expand $\det[g]$ in a Taylor series about $\boldsymbol{\theta}^*$. We can write

$$\delta(\det[g]) = \det[g] + \sum_{i=1}^{n} \frac{\partial \det[g]}{\partial \theta_i}\delta\theta_i + (1/2)\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \det[g]}{\partial\theta_i\,\partial\theta_j}\delta\theta_i\delta\theta_j + ... \tag{26}$$

where all the partial derivatives on the right-hand side are evaluated at $\boldsymbol{\theta}^*$.

At a singularity the first term on the right-hand side, $\det[g]$, is zero by definition. The second term is also zero since $\det[g]$ is also at an extremum for $\boldsymbol{\theta}^*$ – $\det[g]$ is the volume of the ellipsoid and it is always greater than or equal to zero and the minimum is zero. From differential geometry(Millman and Parker 1977), at a *non-singular* point, we have

$$\frac{\partial \det[g]}{\partial \theta_i} = 2\det[g]\Gamma_{ir}^{r} = \sum_{l=1}^{n} \Gamma_{irl} G_{rl} \tag{27}$$

where G_{lr} is the co-factor of g_{lr} in $\det[g]$ and Γ_{ir}^r, Γ_{irl} are the Christoffel symbols(Millman and Parker 1977) of the first and second kind. Since at a singularity, the partial derivatives, $\frac{\partial \det[g]}{\partial\theta_i}$ are zero, some of the Christoffel symbols may be zero. As a consequence some of the terms in the second term of equation (24) may be zero and the accelerations along certain directions may be zero. This is illustrated in detail in the singularity analysis of a general 2R manipulator considered in a later section.

The matrix of second partial derivatives of $\det[g]$ determine the second-order properties at a singular point. If the matrix $\frac{\partial^2 \det[g]}{\partial\theta_i\ \partial\theta_j}$ evaluated at $\boldsymbol{\theta}^*$ has a positive determinant then the volume of the ellipsoid increases from zero. If the matrix has zero determinant, then the volume stays zero and we have to look at higher derivatives of $\det[g]$. Since, $\det[g]$ is product of the eigenvalues of $[g]$, the matrix of second derivatives of $\det[g]$ indicate whether the second partials of zero eigenvalue is positive or zero. It may be noted that a similar analysis with the manipulability measure, $\det[JJ^T]$, have been done by Bedrossian(Bedrossain 1990) for redundant manipulators and he has termed the first type of singularities as escapable or hyperbolic singularities and the second type, with all higher derivatives zero, as inescapable or elliptic singularities.

3.2 Singularity analysis for parallel manipulators

For a parallel manipulator, if the $\det[K^*] \neq 0$, then we can analyze the singularity corresponding to the loss of rank of $[g^*]$ or when $\det[g^*] = 0$. We can replace $[g]$ by $[g^*]$ in the analysis performed for the serial manipulator and can derive analogous results. When $\det[K^*] = 0$, we have a different kind of singularity – a singularity associated with the *gain* of one or more degree-of-freedom(Gosselin and Angeles 1990). This can be seen readily from equation (13), reproduced below for convenience,

$$\mathbf{0} = [K]\dot{\boldsymbol{\theta}} + [K^*]\dot{\boldsymbol{\phi}} \tag{28}$$

We consider the situation, when all the $(n-m)$ actuated joints are locked or $\dot{\boldsymbol{\theta}}$ is set to zero. If $\det[K^*] \neq 0$, all the passive joint rates, $\dot{\boldsymbol{\phi}}$, become zero from equation (28) and as expected we get a structure. From linear algebra, we know that the homogeneous equation, $[K^*]\dot{\boldsymbol{\phi}} = 0$, can have non-trivial solutions (not all $\dot{\phi}_i$ zero) when the matrix $[K^*]$ is singular or $\det[K^*] = 0$. This implies that at the configuration corresponding to loss of rank of $[K^*]$ or when $\det[K^*] = 0$, the structure can have a non-zero $\dot{\phi}_i$, and thereby the structure gains one or more degrees-of-freedom.

A geometric picture of the singularity corresponding to the gain of degree-of-freedom is as follows:

With all the actuated joints locked($\dot{\boldsymbol{\theta}} = 0$), at non-singular positions, we get $\dot{\boldsymbol{\phi}} = 0$ from equation (28). Since $\dot{\boldsymbol{\theta}}$ and $\dot{\boldsymbol{\phi}}$ are both zero, from equation (11), we get, as expected, $\mathbf{v} = 0$. Hence *at a non-singular position* with actuated joints locked, we can think of the velocity distribution as an ellipsoid of zero size. At a singularity, the matrix $[K^*]$ looses rank. If the rank is $(m-1)$ then we can extract the eigenvector of $[K^*]$ corresponding to the zero eigenvalue of $[K^*]$. Let the eigenvector corresponding to the zero eigenvalue be $\dot{\boldsymbol{\phi}}_1$. Since, $c_1\dot{\boldsymbol{\phi}}$ is also an eigenvector with c_1 any scaling constant, from equation (11), we get

$$\mathbf{v} = c_1[J^*]\dot{\boldsymbol{\phi}}_1 \tag{29}$$

and there can be motion along the direction of $[J^*]\dot{\boldsymbol{\phi}}_1$. In this case, we can think of the zero velocity ellipsoid "growing" into a line. If the rank of the matrix $[K^*]$ is $(m-2)$, then with a similar reasoning we can get

$$\mathbf{v} = c_1[J^*]\dot{\boldsymbol{\phi}}_1 + c_2[J^*]\dot{\boldsymbol{\phi}}_2 \tag{30}$$

where $\dot{\boldsymbol{\phi}}_1$, $\dot{\boldsymbol{\phi}}_2$ are the two eigenvectors corresponding to the two zero eigenvalues of $[K^*]$ and c_1, c_2 are the two scaling constants. If we normalize $c_i, i = 1, 2$, to be between -1 and $+1$ (or $c_1^2 + c_2^2 = 1$), then the tip of the velocity vector traces an ellipse[7]. If the rank of $[K^*]$ is $(m-3)$, then the tip of the velocity vector will lie on an ellipsoid. If the rank is less than $(m-3)$, then we have a situation similar to the redundant serial manipulator.

To analyze the second-order properties of the singularities associated with the gain of degree-of-freedom, we consider the acceleration vector with all the actuated joints locked. The acceleration vector is given by

$$\mathbf{a} = \sum_{i=1}^{m} \boldsymbol{\psi}_i \ddot{\phi}_i + \sum_{i,j=1}^{m} \boldsymbol{\psi}_{ij} \dot{\phi}_i \dot{\phi}_j \tag{31}$$

where $\boldsymbol{\psi}_i$, $\boldsymbol{\psi}_{ij}$ are the partial derivatives with respect to the passive joint variables. In addition, we can differentiate equation (28), and on substituting the condition for locked actuated joints, $\ddot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}} = 0$, we get

$$\mathbf{0} = [K^*]\ddot{\boldsymbol{\phi}} + [\dot{K}^*]\dot{\boldsymbol{\phi}} \tag{32}$$

where $[\dot{K}^*]$ is the time derivative of each term of the matrix $[K^*]$.

As pointed out before, at a singularity, the matrix $[K^*]$ looses rank and the second term on the right-hand side is known from the velocity analysis. Equation (32) can only have a non-zero $\ddot{\boldsymbol{\phi}}$ as a solution if the vector $[\dot{K}^*]\dot{\boldsymbol{\phi}}$ lies in the column space of $[K^*]$. Mathematically $\ddot{\boldsymbol{\phi}}$ is non-zero if $\mathbf{y}^T([\dot{K}^*]\dot{\boldsymbol{\phi}}) = 0$ for all $[K^*]^T\mathbf{y} = 0$ and a non-zero $\ddot{\boldsymbol{\phi}}$ can be easily obtained from standard techniques in linear algebra(Golub and Loan 1989). The task space acceleration can be obtained by substituting the non-zero $\ddot{\boldsymbol{\phi}}$ in equation (31).

One special case which ensures $\mathbf{y}^T([\dot{K}^*]\dot{\boldsymbol{\phi}}) = 0$ is when $\dot{\boldsymbol{\phi}}$ is zero. When $\dot{\boldsymbol{\phi}}$ is zero, the velocity $\mathbf{v}$ is also zero with actuator locked. However, the parallel manipulator can have non-zero acceleration since $\ddot{\boldsymbol{\phi}}$ is non-zero.

In the next section, we look at two cases to illustrate the theory developed in the last two sections.

4 Case studies of serial and parallel manipulators

In this section, we illustrate the theory developed in the two previous sections by means of two examples, namely a general two-degree-of-freedom serial manipulator and a three-degree-of-freedom parallel manipulator described in (Lee and Shah 1988).

[7] c_1 and c_2 are similar to $\dot{\theta}_1$ and $\dot{\theta}_2$ in the differential kinematics of serial manipulators, and as in section 2, we can easily prove that the tip of $\mathbf{v}$ lies on an ellipse.

4.1 A general 2R manipulator

Figure 1 shows a two-degree-of-freedom manipulator with two revolute(R) joints of general geometry. The joint variables are θ_1 and θ_2 and the point trajectory is traced by the point $(x, y, z)^T$ in $\Re^3$. Hence the point trajectory is a surface in $\Re^3$. In terms of the link lengths, a_{ij}'s, link offsets, d_i's, and the twists α_{ij}, the mapping function $\boldsymbol{\psi}$ can be written as[8]

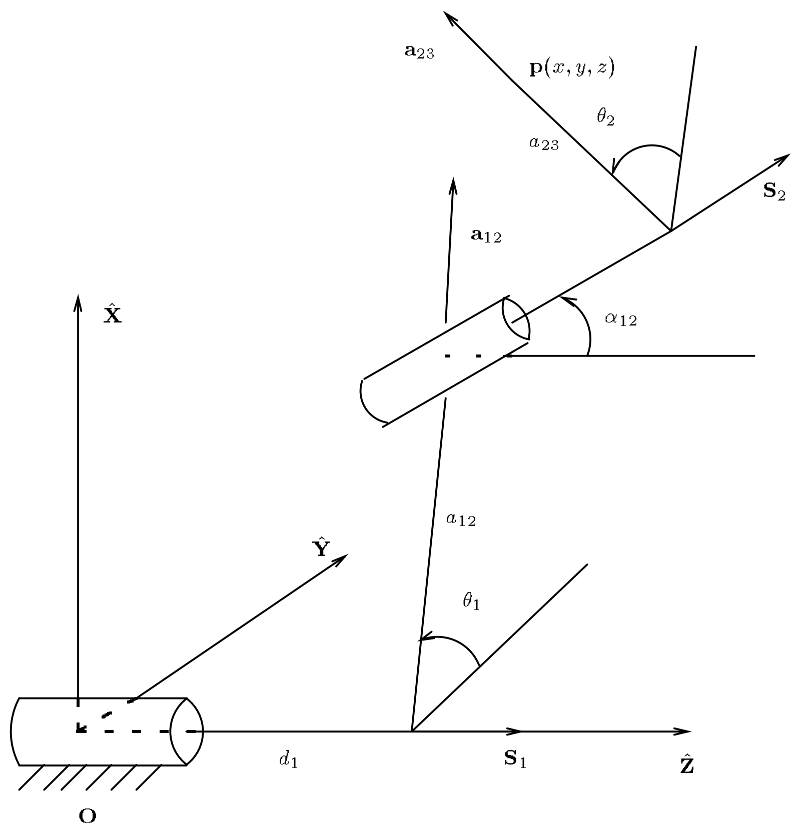

Figure 1: A schematic of a general 2R manipulator

$$(x, y, z)^T = \boldsymbol{\psi}(\theta_1, \theta_2) = d_1\mathbf{S}_1 + a_{12}\mathbf{a}_{12} + d_2\mathbf{S}_2 + a_{23}\mathbf{a}_{23} \tag{33}$$

where

$$\begin{aligned}
\mathbf{S}_1 &= (0,\ 0,\ 1)^T \\
\mathbf{a}_{12} &= (c_1,\ s_1,\ 0)^T
\end{aligned}$$

[8]We will use the symbols c_i, s_i etc. to represent $\cos(\theta_i)$, $\sin(\theta_i)$ etc. respectively throughout the paper.

$$
\begin{aligned}
\mathbf{S}_2 &= (s_1 s\alpha_{12},\ -c_1 s\alpha_{12},\ c\alpha_{12})^T \\
\mathbf{a}_{23} &= [(c_1c_2 - s_1s_2c\alpha_{12}),\ (s_1c_2 + c_1s_2c\alpha_{12}),\ s\alpha_{12}s_2]^T
\end{aligned} \tag{34}
$$

The partial derivatives $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are given by

$$
\begin{aligned}
\boldsymbol{\psi}_1 &= a_{12}(-s_1, c_1, 0)^T + d_2(s\alpha_{12}c_1, s\alpha_{12}s_1, 0)^T + a_{23}(-s_1c_2 - c_1s_2c\alpha_{12}, c_1c_2 - s_1s_2c\alpha_{12}, 0)^T \\
\boldsymbol{\psi}_2 &= a_{23}(-c_1s_2 - s_1c_2c\alpha_{12}, -s_1s_2 + c_1c_2c\alpha_{12}, s\alpha_{12}c_2)^T
\end{aligned} \tag{35}
$$

The coefficients of the metric $[g]$ are given by

$$
\begin{aligned}
g_{11} &= a_{12}^2 + d_2^2 s\alpha_{12}^2 + a_{23}^2(c_2^2 + s_2^2 c\alpha_{12}^2) + 2a_{23}a_{12}c_2 - 2d_2a_{23}c\alpha_{12}s\alpha_{12}s_2 \\
g_{12} &= a_{23}(a_{12}c\alpha_{12}c_2 - d_2 s\alpha_{12}s_2 + a_{23}c\alpha_{12}) \\
g_{22} &= a_{23}^2
\end{aligned} \tag{36}
$$

and

$$
\det[g] = g_{11}g_{22} - g_{12}^2 = a_{23}^2[(a_{12} + a_{23}c_2)^2 s\alpha_{12}^2 + (a_{12}c\alpha_{12}s_2 + d_2 s\alpha_{12}c_2)^2] \tag{37}
$$

It may be noted that the g_{ij}'s and the determinant of $[g]$ are independent of θ_1. This is because the metric coefficients are always independent of translation and rotation of a coordinate system and the effect of θ_1 is equivalent to a rotation of the fixed coordinate system.

The Gaussian curvature is given as

$$
K = \frac{a_{23}s\alpha_{12}^2 c_2 g_{11} g_{22}(a_{12} + a_{23}c_2)}{\det[g]} + s_{12}^2 c_2 a_{23} a_{12} \frac{g_{12}^2}{(\det[g])^2} \tag{38}
$$

At (θ_1, θ_2) given by $(0, 0)$ degrees, the velocity ellipse, in three sectional views and a 3D view, is shown in figure 2. We have assumed $\alpha_{12} = 45^{\circ}$, $a_{12} = d_2 = 1$ and $a_{23} = 1.5$. The maximum and minimum values of the magnitude of velocity for $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$ are $\sqrt{7.9776}$ and $\sqrt{1.0224}$ along the principal axis of the ellipse as shown in figure 2. The ellipse is in the tangent plane with normal along $(0.9285, -0.2626, 0.2626)^T$ and the maximum and minimum velocities are along vectors $(-0.6417, -2.7143, -0.4456)^T$ and $(0.2971, 0.878, -0.9625)^T$ in the tangent plane. The Gaussian curvature at $(0, 0)$ degrees is 0.3092 implying that the point $(0, 0)$ is elliptic. At the point $(0, 120)$ degrees the Gaussian curvature is -0.5791 and the surface is hyperbolic. One can also obtain points where the surface is parabolic.

At a singular point, $\det[g] = 0$ and this implies

$$
\begin{aligned}
a_{12} + a_{23}c_2 = 0 \quad &\text{or} \quad s\alpha_{12} = 0 \\
a_{12}c\alpha_{12}s_2 + d_2 s\alpha_{12}c_2 &= 0
\end{aligned} \tag{39}
$$

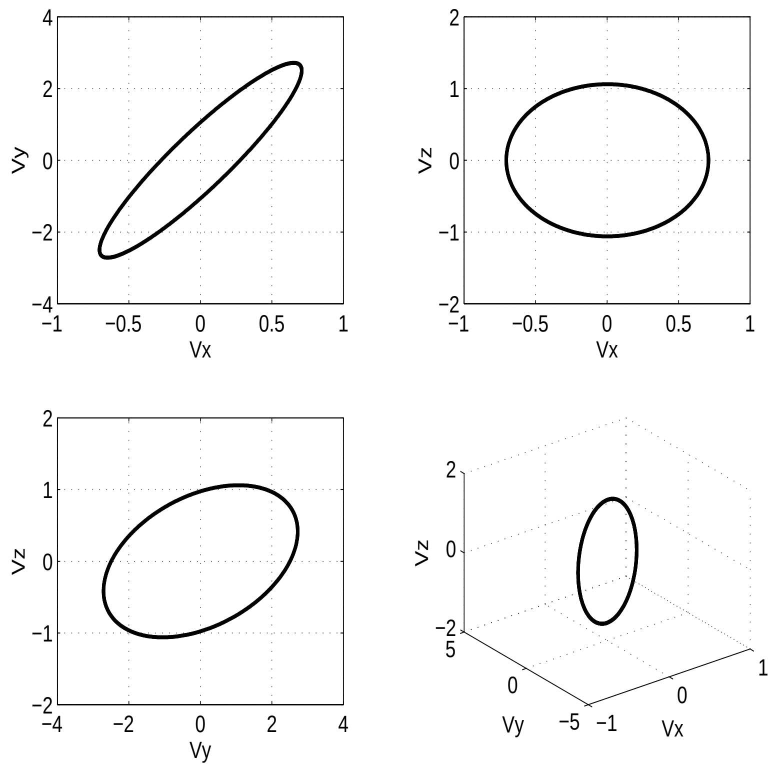

Figure 2: Velocity ellipse at a non-singular point $(\theta_1, \theta_2) = (0, 0)^\circ$

If $s\alpha_{12} = 0$, then the manipulator is planar and the singularities can occur only if $\theta_2 = 0, \pi$. If $\alpha_{12} \neq 0$, then the singularities can occur when

$$\begin{aligned} \tan^2 \alpha_{12} &= \frac{a_{23}^2 - a_{12}^2}{{d_2}^2} \\ \tan(\theta_2/2) &= \frac{d_2 s\alpha_{12}}{a_{23} - a_{12}} \end{aligned} \tag{40}$$

The above equation implies that the general 2R serial manipulator can have singularities only for special values of link lengths, offsets and twist angles, and if one has a geometry satisfying the first equation in (40), the singularities lie along a curve with the value of θ_2 given by the second equation in the set of equations (40).

For the values of $a_{12} = d_2 = 1.0$ and $a_{23} = 1.5$, the singularities occur for $\alpha_{12} = \pm 48.1897°$ and $\theta_2 = \pm 131.8103°$. The velocity ellipse for such values degenerates to a straight line along the unit vector $(-0.7454, -0.4444, -0.4969)^T$. Figure 3 shows the three orthographic views of this line and also a 3D view of the degenerate velocity "ellipse". The maximum velocity for $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$ is 1.5 along the direction of the straight line. By use of equations (40), one can calculate the values of α_{12} and θ_2 for any other set of values of a_{12}, d_2 and a_{23} and get plots as in figure 3.

On substituting α_{12} and θ_2 from equations (40) in expressions for g_{11} and g_{12}, we can show that for the general 2R manipulator $g_{11} = g_{12} = 0$ at a singularity. Since $\det[g]$ is independent of θ_1 all its partial derivatives with respect to θ_1 are zero. In addition, since g_{22} is constant, all the partial derivatives of g_{22} with respect to θ_2 are zero. The partial derivative of $\det[g]$ with respect to θ_2 is given by

$$\frac{\partial \det[g]}{\partial \theta_2} = g_{11}\frac{\partial g_{22}}{\partial \theta_2} + g_{22}\frac{\partial g_{11}}{\partial \theta_2} - 2g_{12}\frac{\partial g_{12}}{\partial \theta_2} \tag{41}$$

At the singularity $\frac{\partial g_{11}}{\partial \theta_2} = 0$ and hence the whole of the right-hand side is zero. From equation (27), we obtain that

$$\sum_{l=1}^{2} (\boldsymbol{\psi}_{ir} \cdot \boldsymbol{\psi}_l) G_{rl} = 0, \quad i, r = 1, 2 \tag{42}$$

Referring to equation (17) and using the above equation, we get at a singularity,

$$\begin{aligned} \mathbf{a} \cdot \boldsymbol{\psi}_1 &= 0 \\ \mathbf{a} \cdot \boldsymbol{\psi}_2 &= g_{22}\ddot{\theta}_2 \end{aligned} \tag{43}$$

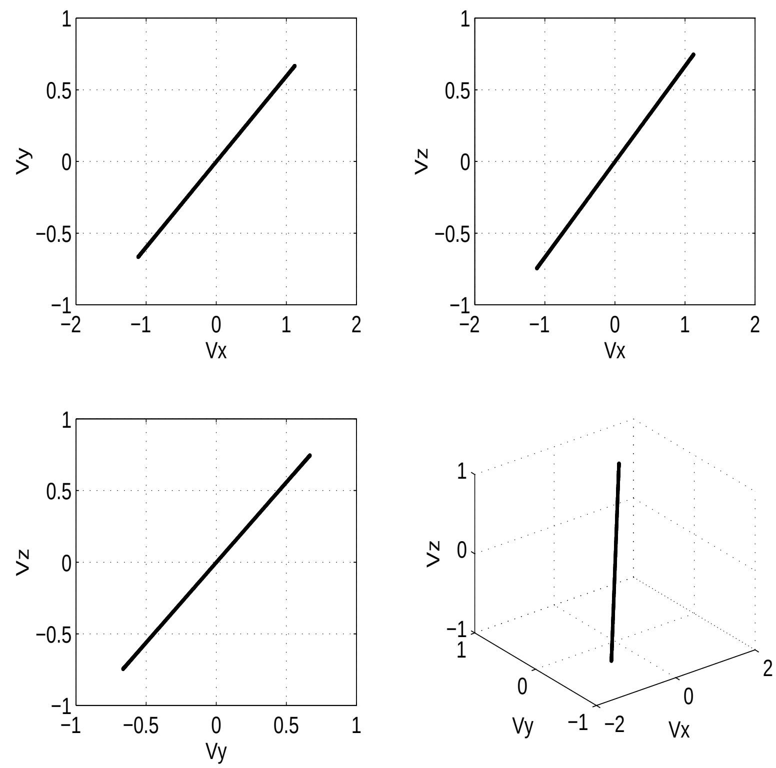

Figure 3: Degenerate velocity "ellipse" at a singular point

The above equation implies that acceleration is only possible along the $\boldsymbol{\psi}_2$ direction at a singularity.

On computing the second partial derivatives of det$[g]$, we find that the only non-zero term is

$$\frac{\partial^2 \det[g]}{\partial \theta_2^2} = g_{22}\frac{\partial^2 g_{11}}{\partial \theta_2^2} = 2a_{23}^4(s_2^2 + c_2^2 c\alpha_{12}^2) \tag{44}$$

The above implies that the singularity in a 2R manipulator is of second order. In addition, the determinant of the matrix of second partial derivatives of det$[g]$ is zero implying that higher-order analysis is required.

It may be noted that in the planar case, $\alpha_{12} = 0$ and $\theta_2 = 0, \pi$, and the first partial derivatives of det$[g]$ are zero. In the second partial derivatives of det$[g]$, the only non-zero term is

$$\frac{\partial^2 \det[g]}{\partial \theta_2^2} = 2a_{23}^2 a_{12}^2 \tag{45}$$

Hence the matrix of second partial derivatives has a zero determinant implying that higher-order analysis is required.

4.2 A RPSSPR-SPR parallel manipulator

In reference (Lee and Shah 1988), the three-loop, three-degree-of-freedom RPSSPR-SPR mechanism of figure 4 has been proposed as a "parallel" wrist. The authors have discussed the direct and inverse kinematics but they have not dealt with its singularities. In this subsection, we use the theory developed in section 3, to analyze the singularities for this parallel manipulator.

The geometry chosen is same as in (Lee and Shah 1988) where the revolute joints axes are assumed to be co-planar and are perpendicular to the medians passing through the respective vertices. Assuming that the length of the medians in the base equilateral triangle are unity, we can obtain the coordinates of the centre of the three spherical joints in the fixed coordinate system $\{0\}$. These are given by

$$\begin{aligned} \mathbf{S}_1 &= [(1 - l_1 c_1),\ 0,\ l_1 s_1]^T \\ \mathbf{S}_2 &= [-0.5(1 - l_2 c_2),\ \sqrt{3}/2(1 - l_2 c_2),\ l_2 s_2]^T \\ \mathbf{S}_3 &= [-0.5(1 - l_3 c_3),\ -\sqrt{3}/2(1 - l_3 c_3),\ l_3 s_3]^T \end{aligned} \tag{46}$$

where $\theta_i,\ i = 1, 2, 3$ are rotations at the three passive rotary joints and $l_i,\ i = 1, 2, 3$ are the translations at the actuated prismatic joints.

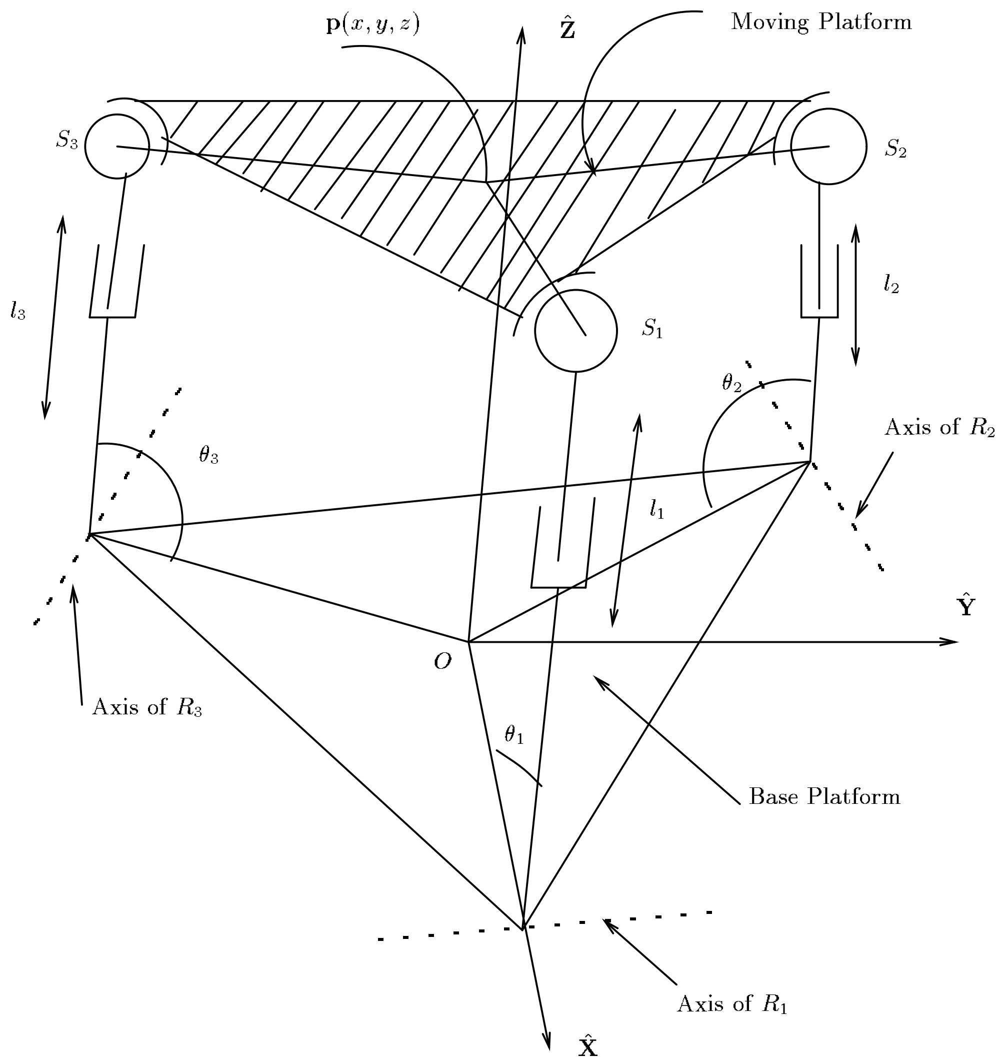

Figure 4: The RPSSPR-SPR parallel manipulator

The loop closure equations are obtained from the fact that the distance between the spherical joints are constant and are of the form

$$(\mathbf{S}_i - \mathbf{S}_j) \cdot (\mathbf{S}_i - \mathbf{S}_j) = k_{ij}^2, \qquad i, j = 1, 2, 3, i \neq j \tag{47}$$

where k_{ij} is the distance between spherical joint i and spherical joint j respectively.

Differentiating the three constraint equations with respect to time, we get

$$\begin{aligned}
\mathbf{0} \;=\; & \begin{pmatrix} 3l_1s_1 - l_1l_2s_1c_2 - 2l_1l_2c_1s_2 \\ 0 \\ 3l_1s_1 - l_1l_3s_1c_3 - 2l_1l_3c_1s_3 \end{pmatrix} \dot{\theta}_1 + \begin{pmatrix} 3l_2s_2 - l_1l_2c_1s_2 - 2l_1l_2s_1c_2 \\ 3l_2s_2 - l_2l_3s_2c_3 - 2l_2l_3c_2s_3 \\ 0 \end{pmatrix} \dot{\theta}_2 + \\
& \begin{pmatrix} 0 \\ 3l_3s_3 - l_2l_3c_2s_3 - 2l_2l_3s_2c_3 \\ 3l_3s_3 - l_1l_3c_1s_3 - 2l_1l_3s_1c_3 \end{pmatrix} \dot{\theta}_3 + \begin{pmatrix} 2l_1 - 3c_1 + l_2c_1c_2 - 2l_2s_1s_2 \\ 0 \\ 2l_1 - 3c_1 + l_3c_1c_3 - 2l_3s_1s_3 \end{pmatrix} \dot{l}_1 + \\
& \begin{pmatrix} 2l_2 - 3c_2 + l_1c_1c_2 - 2l_1s_1s_2 \\ 2l_2 - 3c_2 + l_3c_2c_3 - 2l_3s_2s_3 \\ 0 \end{pmatrix} \dot{l}_2 + \begin{pmatrix} 0 \\ 2l_3 - 3c_3 + l_2c_2c_3 - 2l_2s_2s_3 \\ 2l_3 - 3c_3 + l_1c_1c_3 - 2l_1s_1s_3 \end{pmatrix} \dot{l}_3
\end{aligned} \tag{48}$$

The above equation can be written in the form of equation (13) as

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = -[K^*]^{-1}[K] \begin{pmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{l}_3 \end{pmatrix} \tag{49}$$

where the columns of $[K^*]$ and $[K]$ are coefficients of $\dot{\theta}_i,\ i = 1, 2, 3$ and $\dot{l}_i,\ i = 1, 2, 3$ respectively.

Assuming all the lengths k_{ij}'s are $\sqrt{3}/2$(the lengths of the medians of the top platform are 0.5 units each) the coordinates of the centroid of the moving platform are given as

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= (1/3)(\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3) \\
&= \frac{1}{3} \left[\begin{pmatrix} 1 - l_1c_1 \\ 0 \\ l_1s_1 \end{pmatrix} + \begin{pmatrix} (-1/2)(1 - l_2c_2) \\ (\sqrt{3}/2)(1 - l_2c_2) \\ l_2s_2 \end{pmatrix} + \begin{pmatrix} (-1/2)(1 - l_3c_3) \\ (-\sqrt{3}/2)(1 - l_3c_3) \\ l_3s_3 \end{pmatrix} \right]
\end{aligned} \tag{50}$$

and the velocity of the centre is given by

$$\mathbf{v} \;=\; (1/3) \left[\begin{pmatrix} l_1s_1 \\ 0 \\ l_1c_1 \end{pmatrix} \dot{\theta}_1 + \begin{pmatrix} (-1/2)l_2s_2 \\ (\sqrt{3}/2)l_2s_2 \\ l_2c_2 \end{pmatrix} \dot{\theta}_2 + \begin{pmatrix} (-1/2)l_3s_3 \\ (-\sqrt{3}/2)l_3s_3 \\ l_3c_3 \end{pmatrix} \dot{\theta}_3 \right]$$

$$+(1/3)\left[\begin{pmatrix} -c_1 \\ 0 \\ s_1 \end{pmatrix}\dot{l}_1 + \begin{pmatrix} (1/2)c_2 \\ (-\sqrt{3}/2)c_2 \\ s_2 \end{pmatrix}\dot{l}_2 + \begin{pmatrix} (1/2)c_3 \\ (\sqrt{3}/2)c_3 \\ s_3 \end{pmatrix}\dot{l}_3\right]$$

$$= [J^*]\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} + [J]\begin{pmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{l}_3 \end{pmatrix} \tag{51}$$

where $[J^*]$ and $[J]$ are 3×3 matrices obtained from the coefficients of $\dot{\theta}_i,\ i = 1, 2, 3$, and $\dot{l}_i,\ i = 1, 2, 3$ respectively. Using equation (49) in equation (51), we get

$$\mathbf{v} = ([J] - [J^*][K^*]^{-1}[K])(\dot{l}_1, \dot{l}_2, \dot{l}_3)^T = \sum_{i=1}^{3} \alpha_i \dot{l}_i \tag{52}$$

The metric coefficients in this case are the elements of the matrix $([J]-[J^*][K^*]^{-1}[K])^T([J]-[J^*][K^*]^{-1}[K])$, and the three-degree-of-freedom RPSSPR-SPR parallel manipulator will *loose* one or more degree-of-freedom when

$$\det[g^*] = \det[([J] - [J^*][K^*]^{-1}[K])^T([J] - [J^*][K^*]^{-1}[K])] = 0 \tag{53}$$

The above equation is a function of the passive variables $\theta_i,\ i = 1, 2, 3$ and the three actuated variables $l_i,\ i = 1, 2, 3$. Equation (53) together with the loop closure equations (47) represent 4 equations in 6 unknowns and hence the singularities occur on a high-order 2D surface. It is very difficult to derive analytical results for this case; we, therefore, present numerical results.

At a typical non-singular point given by $(l_1, l_2, l_3) = (0.5, 1.0, 2.0)$ meters, and the corresponding passive variables,$(\theta_1, \theta_2, \theta_3)$, given by $(0.4, 0.7535, 0.2402)$ radians, the tip of the velocity vector will lie on the ellipsoid shown in figure 5. The maximum, intermediate, and minimum velocities along the principal axes of the ellipsoid are given by $0.3724, 0.3162, 0.2031$ m/sec respectively. The directions of the corresponding principal axes are $(0.9921, -0.0394, 0.1187)^T$, $(0.1166, 0.6338, -0.7646)^T$ and $(-0.0452, 0.7724, 0.6335)^T$ respectively.

From numerical solution of the constraint equations and the condition for loss of degree-of-freedom, we find that the leg lengths, (l_1, l_2, l_3), given by $(0.5, 1.0, 1.9710)$ meters and the corresponding passive variables, $(\theta_1, \theta_2, \theta_3)$, given by $(1.1691, 0.4781, 0.2355)$ radians is a singular point. The tip of the velocity ellipsoid no longer lies on an ellipsoid and the eigenvalues of the matrix $([J] - [J^*][K^*]^{-1}[K])$ are $(0.7647, 0, 2.2773)$ m/sec. At this singular point, the mechanism looses one degree-of-freedom and the velocity distribution is the ellipse

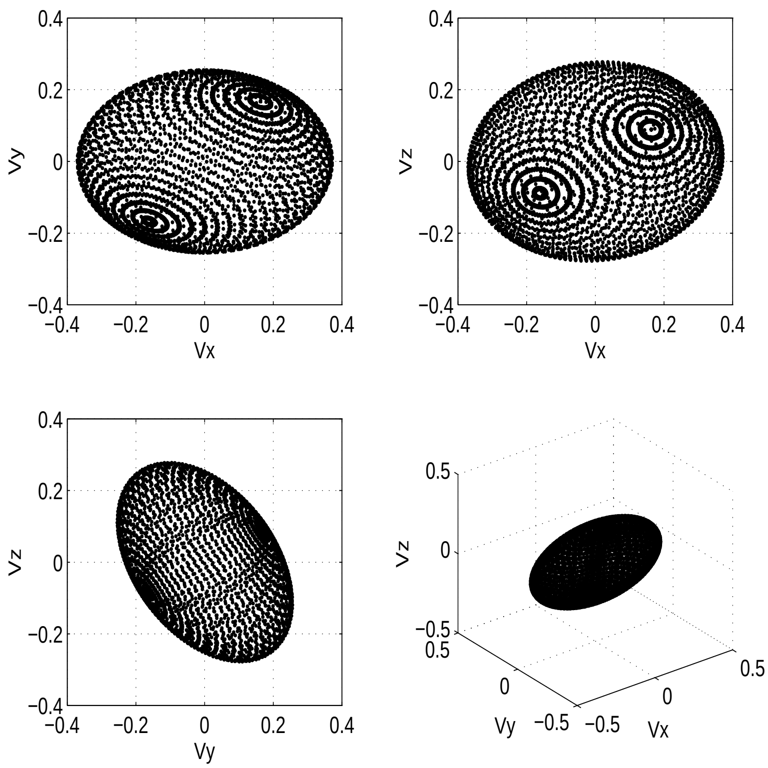

Figure 5: Velocity ellipsoid at a non-singular point

shown in sectional views and as a 3D plot in figure 6. The centroid of the top platform can move along any direction in a plane spanned by the vectors corresponding to the two non-zero eigenvalues.

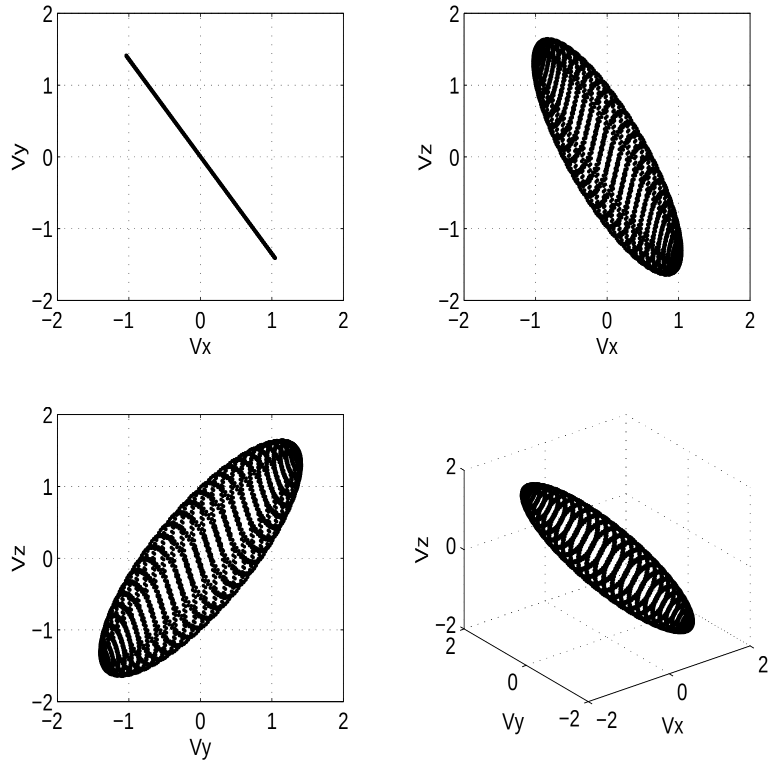

Figure 6: Velocity ellipse at a singular point

The RPSSPR-SPR parallel manipulator will *gain* one or more degrees-of-freedom when

$$\begin{aligned}\det[K^*] \;=\;& (3l_1s_1 - l_1l_2s_1c_2 - 2l_1l_2c_1s_2)\times(3l_2s_2 - l_2l_3s_2c_3 - 2l_2l_3c_2s_3)\times\\ & (3l_3s_3 - l_1l_3c_1s_3 - 2l_1l_3s_1c_3)\\ & +(3l_1s_1 - l_1l_3s_1c_3 - 2l_1l_3c_1s_3)\times(3l_2s_2 - l_1l_2c_1s_2 - 2l_1l_2s_1c_2)\times\\ & (3l_3s_3 - l_2l_3c_2s_3 - 2l_2l_3s_2c_3) = 0 \qquad (54)\end{aligned}$$

The above equation is a function of all the passive and active joint variables and again together with the loop closure equation (47) represent a set of 4 equations in 6 variables. Thus the singularities resulting in a *gain* of one or more degrees-of-freedom also lie on a 2D surface. It is very difficult to get analytical expressions for this surface and we present numerical results.

At the values of leg-lengths, (l_1, l_2, l_3), given by $(0.575, 0.483, 0.544)$ meters respectively and the corresponding passive variables, $(\theta_1, \theta_2, \theta_3)$, given by $(-0.3441, -0.0138, 0.2320)$ radians, $\det[K^*]$ is found to be very close to zero. The eigenvalues of $[K^*]$ are -0.5565, 0 and 0.4509 respectively and the three eigenvectors corresponding to the three eigenvalues are $(-0.8098, 0.3571, -0.4656)^T$, $(-0.3109, -0.8743, -0.3727)^T$ and $(-0.0877, -0.4781, -0.8739)^T$ respectively. Hence at this point, the mechanism gains one degree-of-freedom and the velocity of the centroid, with all actuated joints locked, is given as

$$\mathbf{v} = \begin{pmatrix} -0.0647 \\ 0 \\ 0.1804 \end{pmatrix} \dot{\theta}_1 + \begin{pmatrix} 0.0011 \\ -0.0019 \\ 0.1610 \end{pmatrix} \dot{\theta}_2 + \begin{pmatrix} -0.0208 \\ -0.0361 \\ 0.1763 \end{pmatrix} \dot{\theta}_3 \tag{55}$$

where $(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)^T$ is the eigenvector $\alpha \times (-0.3109, -0.8743, -0.3727)^T$ with α arbitrary. It is clear that the velocity vector lies along a straight line and the mechanism has *gained* instantaneously a degree-of-freedom at this singular point. Figure 7 shows the velocity distribution at the singular point.

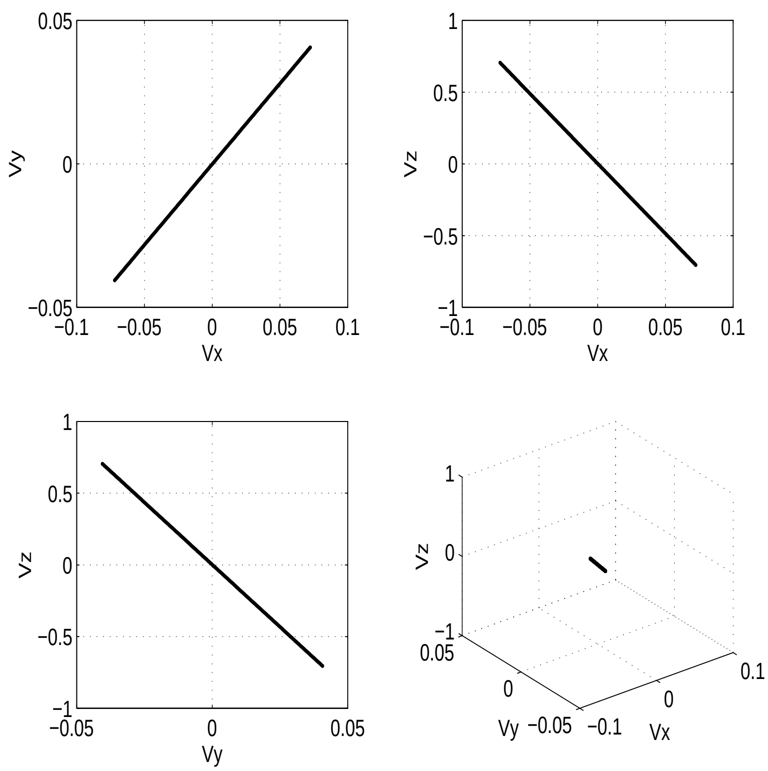

Figure 7: Velocity at a singular point

At the values of leg-lengths, (l_1, l_2, l_3), given by $(1.9363, 2.9998, 1.9363)$ meters respectively and the corresponding passive variables, $(\theta_1, \theta_2, \theta_3)$, given by $(1.3096, 0.9817, 1.3096)$

radians, $\det[K^*]$ is also found to be very close to zero. The eigenvalues of $[K^*]$ are 0, 0, 2.8058 respectively. Hence at this configuration the mechanism gains two degrees of freedom. The velocity distribution, in this case an ellipse, is shown in figure 8. The singularities corresponding to gain of two degrees of freedom lie on a curve in $\Re^3$.

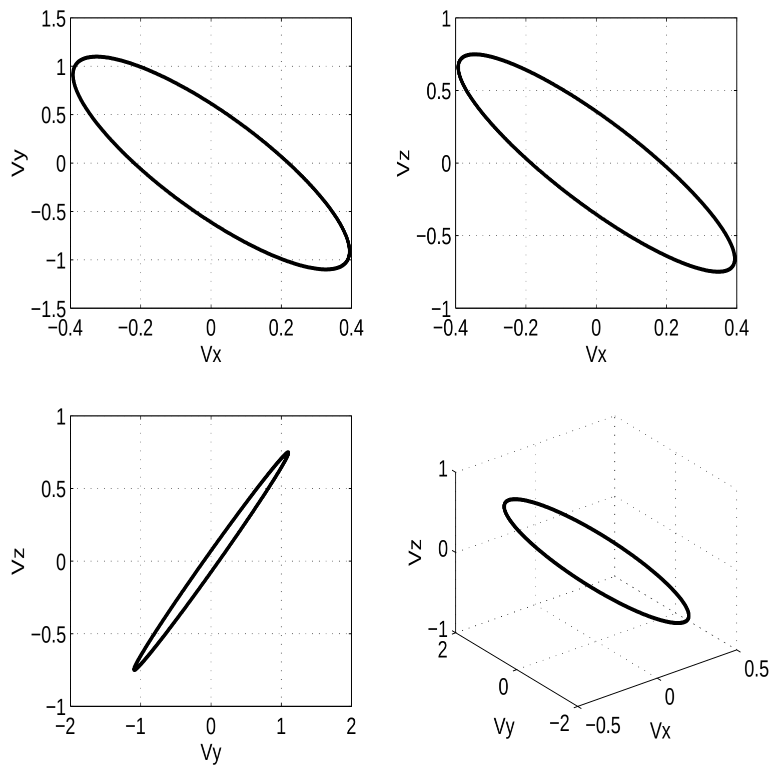

Figure 8: Velocity ellipse at a singular point

Finally, at values of leg-lengths, (l_1, l_2, l_3), given by $(0.5, 0.5, 0.5)$ meters respectively and the corresponding passive variables, $(\theta_1, \theta_2, \theta_3)$, given by $(0, 0, 0)$ radians, the eigenvalues $\det[K^*]$ are found to be all zero. Hence at this singular configuration, the mechanism gains three degrees of freedom. The velocity distribution, in this case would be an ellipsoid. Geometrically, at this configuration, the moving platform is in the same plane as the base platform and one can show that there are 8 such possible configurations with $l_i = 0.5$ or 2.0 with all θ_i as zero.

5 Conclusion

In this paper, we have presented a general geometric framework for differential analysis of point trajectories traced out by multi-degree-of-freedom serial and parallel, non-redundant manipulators. At non-singular points, the tip of the velocity vector of a point on the end-effector lies on an ellipsoid or ellipse. At singular configurations, the ellipsoid degenerates to an ellipse, a line or a point depending on the number of degrees-of-freedom lost at that point. For a parallel manipulator, at a gain of degree-of-freedom singularity, there is a growth to a line, an ellipse or an ellipsoid depending on the number of degrees-of-freedom gained. In both serial and parallel manipulators, the partial derivatives of the metric coefficients and the rate of change of shape and size of the ellipsoid or ellipse can be used to determine the possible accelerations at the singular configurations. The developed theory was illustrated with the help of a serial and a parallel manipulator examples.

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