Modeling of Axially Translating Flexible Beams

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Abstract

The axially translating flexible beam with a prismatic joint can be modeled using the Euler-Bernoulli beam equation together with the convective terms. In general, the method of separation of variables cannot be applied to solve this partial differential equation. In this paper, we present a non-dimensional form of the Euler-Bernoulli beam equation using the concept of group velocity and present conditions under which separation of variables and assumed modes method can be used. The use of clamped-mass boundary conditions lead to a time-dependent frequency equation for the translating flexible beam. We present a novel method to solve this time-dependent frequency equation by using a differential form of the frequency equation. The assumed mode/Lagrangian formulation of dynamics is employed to derive closed form equations of motion. We show using the Lyapunov's first method that the dynamic response of flexural modal variables become unstable during retraction of the flexible beam, compared to the stable dynamic response during extension of the beam. Numerical simulation results are presented for the uniform axial motion induced transverse vibration for a typical flexible beam.

1. INTRODUCTION

Modeling issues associated with flexible beams in translational motion and with a prismatic joint at one end have been receiving attention lately [1 – 6]. When a beam with the prismatic joint is modeled as flexible, the system becomes a moving boundary value problem as the spatial domain of the system changes with time. Moving boundary value problems have also been considered in the deployment dynamic analysis of flexible appendages of a spacecraft [7, 8].

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Tabarrok et al.[1] studied the dynamics of an axially moving beam. They presented properties of the mode shapes of clamped-free beams in flexure, as the beam length varies with time. They also derived the equations of motion of a simple cantilever beam having an axial motion on a stationary rigid base by using Newton's second law. Buffinton and Kane[2] studied the dynamics of a beam moving at a prescribed rate over two bilateral supports. Regarding the supports as kinematical constraints imposed on an unrestrained beam, equations of motion were formulated by using an alternative form of the Kane's method[9] and using assumed modes technique to discretize the beam. Wang and Wei[3] studied the vibration problem of a moving slender prismatic beam using a Galerkin approximation with time-dependent basis functions and by applying Newton's second law. Yuh and Young[4] presented the experimental results to validate the approximated dynamic model derived using assumed modes method for a flexible beam which has a rotational and translational motion. Stylianou and Tabarrok[6] developed governing equations of motion for axially moving beams through finite element discretization. They have used variable-domain beam element wherein the number of elements used in the model remains fixed, while their sizes change with time in a prescribed manner. Tsuchiya[8] studied the dynamics of a spacecraft during extension of flexible appendages under the assumption of small extension velocity. Extensive discussions about this assumption were made by Jankovic[10].

In all the aforementioned works, it is invariably assumed that the translating flexible beams can be modeled as Euler-Bernoulli beams in flexure with clamped-free boundary conditions, leading to a time-independent frequency equation [2, 5]. The 'free' boundary condition however may lead to inaccurate mode shapes and over-estimated eigen frequencies which may have destabilizing effect when the axially moving elastic beam carries a finite load at the distal end[11]. In such cases the 'clamped-mass' boundary conditions are more appropriate. More importantly, the applicability of assumed modes method to discretize flexibility of a translating elastic beam may not be valid, as the principle of separation of space-dependent eigenfunctions and time-dependent modal amplitudes is not valid under general conditions. Therefore it is imperative to derive conditions under which the assumed modes method can be used, as this method has some technical advantages[11] over the other methods for modeling and solving the continuous dynamical systems.

In the analysis of dynamical systems, it is important to discuss the stability properties for different motion characteristics of the system. Elmaraghy and Tabarrok[12] discussed the dynamic stability of an axially accelerated elastic beam between two encastré ends subject to a prescribed periodic root force. They reported regions of instability of this system for various combinations of the excitation frequency and amplitude of the axial oscillations. In their investigation, although the beam under study had axial motion, the vibrating length of the beam remained constant due to the fixed supports at both the ends. Zajaczkowski and his co-workers[13, 14] later determined the instability regions of an axially moving beam subject to periodic input force with varying vibrating length of the beam. Stylianou and Tabarrok[6] investigated the dynamic stability characteristics of a flexible extendible beam for various extrusion profiles. In a different study, Wang and Wei[3] have reported stability results for a translating beam with the prismatic joint. It is significant to note that their results are in contrast to our results in this paper, however, our results are consistent with the recent results reported by Stylianou and Tabarrok[6].

In this paper, we present a discussion on the applicability of using separation of variables for a translating flexible beam. We present the notion of group velocity for dispersive waves and a non-dimensionalized Euler-Bernoulli beam equation based on this group velocity. We show that if the beam is translating at a constant, slow (compared to the group velocity) speed, the assumed modes method can be used. The use of clamped-mass boundary conditions lead to a time-dependent frequency equation for the translating flexible beam. We present a novel method to solve this time-dependent frequency equation by using a differential form of the transcendental frequency equation. We then derive the closed form dynamic equations of motion for the translating flexible beam using the Lagrangian formulation in conjunction with the assumed modes method. We show using the Lyapunov's first method that the dynamic response of flexural modal variables become unstable during retraction of the flexible beam, compared to the stable dynamic response during extension of the beam. Finally, we present dynamic simulation results and show that they agree with an FEM based simulation.

2. MODELING OF A TRANSLATING FLEXIBLE BEAM

Figure 1 shows an uniform flexible beam, of length $l(t)$ outside the rigid hub at time t, vibrating in the Z-X plane and moving axially at a specified velocity $U(t)$ along the Z direction. The portion of the beam to the left of origin of the fixed coordinate system $(l^0 - l(t))$, where l^0 is the total length of the beam, is assumed not to be vibrating. Any arbitrary material point along the neutral axis of the beam is located by s. We assume that the beam is inextensible along its neutral axis and hence the axial velocity $U(t)$ is independent of s[1]. Denoting the elastic displacement of the point with reference to the

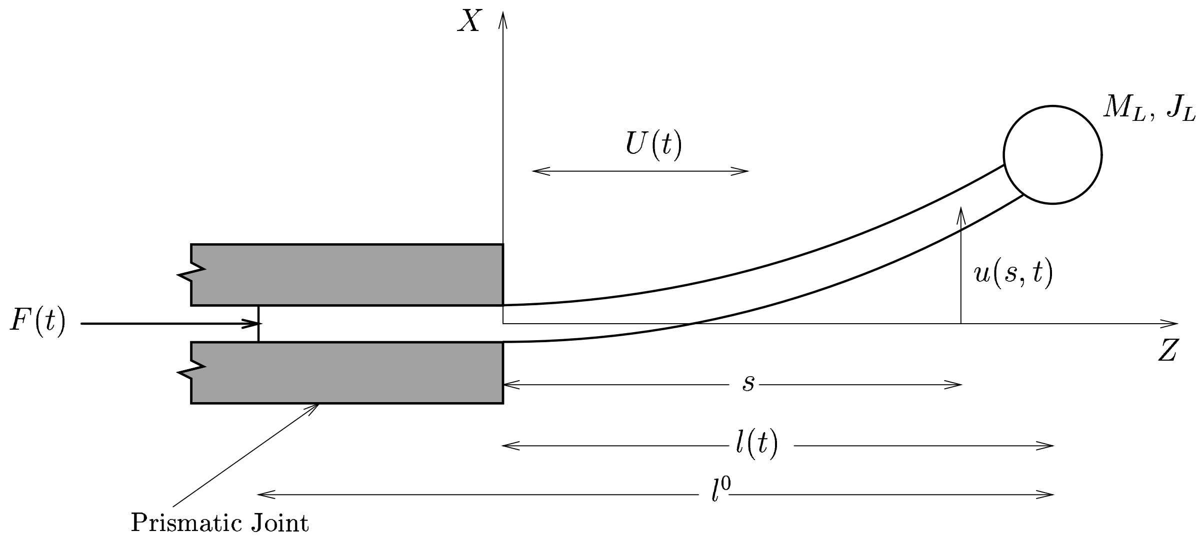

Figure 1: A schematic of the prismatic jointed flexible link with clamped and end-mass conditions

neutral axis at time t by $u(s,t)$, we can write the free vibration equation of the beam using the Euler-Bernoulli beam theory (neglecting the shear deformation and rotary inertia effects [15]) as,

$$EI\frac{\partial^4 u(s,t)}{\partial s^4} + \rho A\left(\frac{\partial^2 u(s,t)}{\partial t^2} + 2U\frac{\partial^2 u(s,t)}{\partial s \partial t} + U^2\frac{\partial^2 u(s,t)}{\partial s^2} + \frac{dU}{dt}\frac{\partial u(s,t)}{\partial s}\right) = 0 \qquad (1)$$

where $s \in (0, l(t))$, EI is the flexural rigidity, ρ is the density of the material and A is the cross-sectional area of the beam. It should be noted in general that the right-hand side of equation(1) will have the axial root force in the Z direction (see equation(11) of [1], and equation(19) of [4]). The boundary conditions for the above partial differential equation(1), for the clamped-mass case are

$$\begin{aligned}
[u(s,t)]_{s=0} &= 0 & EI\left[\frac{\partial^2 u(s,t)}{\partial s^2}\right]_{s=l(t)} &= -J_L\left[\frac{\partial^3 u(s,t)}{\partial t^2 \partial s}\right]_{s=l(t)} \\
\left[\frac{\partial u(s,t)}{\partial s}\right]_{s=0} &= 0 & EI\left[\frac{\partial^3 u(s,t)}{\partial s^3}\right]_{s=l(t)} &= M_L\left[\frac{\partial^2 u(s,t)}{\partial t^2}\right]_{s=l(t)}
\end{aligned} \qquad (2)$$

where M_L and J_L are the concentrated mass and rotary inertia of the load at the tip of the beam (see Figure 1).

The above equation(1) contains the convective terms $2U\frac{\partial^2 u(s,t)}{\partial s \partial t}$, $U^2\frac{\partial^2 u(s,t)}{\partial s^2}$, and $\frac{dU}{dt}\frac{\partial u(s,t)}{\partial s}$, and if the axial velocity(U) is zero, it reduces to the standard Euler-Bernoulli beam equation with clamped-mass boundary conditions. The above equation also represents a moving boundary value problem as the domain governed by this equation changes with time. In this most general form, the above partial differential equation cannot be solved using the separation of variables method, as this method requires the general shape of the beam displacement not to change with time, while only the amplitude of this shape to change with time [16]. Therefore, it is required that the numerical solution to the partial differential equation(1) will have to be determined by either using finite difference or finite element based schemes. It should be noted that these numerical schemes are however computationally very expensive for a specified numerical accuracy and special programming considerations are necessary. Moreover, as the finite element model uses polynomial mode shape functions which do not belong to the class of *complete* set of functions, monotonic convergence to actual solution cannot always be guaranteed [11, 16]. However, convergence can be improved by considering large number of elements in the model [6]. In the rest of the section we present conditions under which the separation of variables and the assumed modes method can be used to solve the above problem.

Let us introduce the non-dimensional variables: $\eta = s/l^0$, and $\tau = t/(l^0/U_g)$ with $U_g = \frac{1}{l^0}\sqrt{\frac{EI}{\rho A}}$. Note that η, τ, and U_g are based on the fully extended length of the beam l^0, as this would give the worst[1] case. The quantity U_g is the "group velocity" of the dispersive waves of the Euler-Bernoulli beam equation[17] and l^0/U_g denotes the time taken for a disturbance induced at one end of the beam to travel the entire length of beam l^0. We observe that for a rigid body ($EI \to \infty$), the time(l^0/U_g) taken for the disturbance to travel over the entire domain(l^0) approaches zero. On the other hand, for a highly flexible beam (EI is small) or for a very long beam (l^0 is large), the time for the disturbance to travel the entire domain will be large. It may be mentioned that for times smaller than l^0/U_g the vibratory motion of the beam is *not* governed by the equation(1). For times much greater than l^0/U_g however, the vibratory motion of the translating beam is completely governed by the equation(1) and one can use the instantaneous mode shapes of the cantilever beam for discretization of the continuum model under certain conditions.

Rewriting the partial differential equation in terms of the non-dimensional variables,

[1]We use the non-dimensional U/U_g and hence the worst case is when U_g is small, i.e. when l^0 is used.

η, τ and the ratio U/U_g, we get

$$\frac{\partial^4 u(\eta,\tau)}{\partial \eta^4}+\frac{\partial^2 u(\eta,\tau)}{\partial \tau^2}+2\left(\frac{U}{U_g}\right)\frac{\partial^2 u(\eta,\tau)}{\partial \eta \partial \tau}+\left(\frac{U}{U_g}\right)^2\frac{\partial^2 u(\eta,\tau)}{\partial \eta^2}+\left(\frac{d}{d\tau}\left(\frac{U}{U_g}\right)\right)\frac{\partial u(\eta,\tau)}{\partial \eta}=0 \quad (3)$$

with the clamped-mass boundary conditions:

$$\begin{aligned} [u(\eta,\tau)]_{\eta=0} &= 0 & \left[\frac{\partial^2 u(\eta,\tau)}{\partial \eta^2}\right]_{\eta=l/l^0} &= -\frac{J_L}{\rho A {l^0}^3}\left[\frac{\partial^3 u(\eta,\tau)}{\partial \tau^2 \partial \eta}\right]_{\eta=l/l^0} \\ \left[\frac{\partial u(\eta,\tau)}{\partial \eta}\right]_{\eta=0} &= 0 & \left[\frac{\partial^3 u(\eta,\tau)}{\partial \eta^3}\right]_{\eta=l/l^0} &= \frac{M_L}{\rho A l^0}\left[\frac{\partial^2 u(\eta,\tau)}{\partial \tau^2}\right]_{\eta=l/l^0} \end{aligned} \quad (4)$$

We can make the following observations from the above equations(3-4):

1. The coefficients of the first two terms are unity and the third, fourth and fifth terms are in terms of U/U_g and the derivative of U/U_g with respect to τ. For a constant axial velocity, the term containing the derivative of U/U_g with respect to τ is zero. For the third and fourth terms to be dominant, U/U_g should be large.

2. The ratio U/U_g, for a given U, is largest for the smallest U_g. The smallest U_g is obtained when the beam is fully extended, i.e. when $l = l^0$.

3. If $U/U_g \ll 1$, then by dimensional analysis we can neglect the third and fourth terms. In typical simulations and experiments, see for example in Yuh and Young[4], U is approximately 0.1 m/sec, and U_g is approximately 3.03 m/sec giving $U/U_g \approx 0.033 \ll 1$. Hence the convective terms can be neglected. We have run simulations with various axial speeds U, upto 1 m/sec with $U_g = 59.34$ m/sec, and have observed that the contribution of the convective terms are much smaller compared to the first two terms. In particular, the tip deflections obtained using the instantaneous mode shapes, after neglecting the convective terms match quite accurately with those obtained from a FEM based model derived from the complete partial differential equation[11].

4. Once the convective terms are dropped, we are left with the standard Euler-Bernoulli beam equation for a clamped-mass cantilever. Separation of variables or the assumed modes method can then be used with the eigen-frequencies based on the fully extended length l^0. Even if the length is changing continuously with time(slowly compared to U_g) we can still assume that the "instantaneous eigen-modes" i.e., the mode shapes of the translating beam at every instant of time approximated by that of a cantilever beam[4], can be used. However, we have to solve for the slowly and continuously

changing "eigen-frequencies" (often called 'quasi-frequencies' [10]) at each instant of time. This procedure is illustrated below.

Let the lateral deflection be described as $u(\eta,\tau) = \boldsymbol{\psi}(\eta)^T\boldsymbol{\xi}(\tau)$, where spatial admissible functions $\boldsymbol{\psi}(\eta) \triangleq [\psi_1(\eta), \psi_2(\eta), \ldots, \psi_n(\eta)]^T$ are the *complete* eigen functions (in the sense that for arbitrary square-integrable $u(\eta,\tau)$: $\min \| u(\eta,\tau) - \boldsymbol{\psi}(\eta)\boldsymbol{\xi}(\tau) \|^2 = 0$). Then following the standard procedure of assumed modes method, we can write

$$\frac{d^4\psi_i(\eta)}{d\eta^4}\xi_i(\tau) + \psi_i(\eta)\frac{d^2\xi_i(\tau)}{d\tau^2} = 0 \qquad i = 1, 2, \ldots, n \tag{5}$$

For separation of variables, the ratio $(\frac{d^2\xi_i(\tau)}{d\tau^2})/\xi_i(\tau) = -(\frac{d^4\psi_i(\eta)}{d\eta^4})/\psi_i(\eta)$ must be constant [16]. This constant is usually denoted by $-\omega_i^2$ and are called the 'eigen-frequencies' of the system(5). The eigen-frequencies are related to the roots(β_i) of the frequency equation (also called the 'wave number' of system(5)) by the dispersion relation[17], $\omega_i = \frac{U_g}{l}\beta_i^2$. It can be seen that when length of the vibrating beam(l) changes continuously with time, these eigen-frequencies will also change continuously with time irrespective of β_i which are determined by the end-conditions.

5. We observe from the boundary conditions (equation(4)) that it is reasonable to use "free" end-conditions for the choice of eigen functions($\boldsymbol{\psi}(\eta)$) only if $\frac{J_L}{\rho A l^3} \ll 1$ and $\frac{M_L}{\rho A l} \ll 1$, when the beam is fully extended (i.e. $l = l^0$). On the other hand, if the rotary inertia(J_L) and mass(M_L) of the load are comparable to that of the vibrating beam, it is more appropriate and correct to use the "mass" end-conditions. The mass end-conditions lead to time-dependent frequency equation as shown in the following.

   The eigen functions $\boldsymbol{\psi}(\eta)$ satisfying the "clamped-mass" boundary conditions are given by,

$$\psi_i(\eta) = C_i\,[\,\cos(\beta_i\eta) - \cosh(\beta_i\eta) + \nu_i\,(\sin(\beta_i\eta) - \sinh(\beta_i\eta))\,] \tag{6}$$

   where,

$$\nu_i = \frac{\sin\beta_i - \sinh\beta_i + M\beta_i(\cos\beta_i - \cosh\beta_i)}{\cos\beta_i + \cosh\beta_i - M\beta_i(\sin\beta_i - \sinh\beta_i)} \tag{7}$$

   and β_i are solutions of the frequency equation,

$$\begin{aligned}&(1 + \cosh\beta_i\cos\beta_i) - M\beta_i(\cosh\beta_i\sin\beta_i - \sinh\beta_i\cos\beta_i)\\ &- J{\beta_i}^3(\cosh\beta_i\sin\beta_i + \sinh\beta_i\cos\beta_i) + MJ{\beta_i}^4(1 - \cosh\beta_i\cos\beta_i) = 0\end{aligned} \tag{8}$$

where $M = \frac{M_L}{\rho A l}$, $J = \frac{J_L}{\rho A l^3}$. The C_i are constants which may be chosen to normalize the eigen functions. It can be seen from equation(8) that when 'clamped-free' end-conditions are used the roots(β_i) of the equation will be constants, however with the 'clamped-mass' end-conditions they will change with time, albeit slowly.

6. The time dependent frequency equation can be solved by either using a root finding algorithm at each instant of time or by using a "table look-up" approach. The former approach may lead to considerable increase in computational time, while the latter requires a large storage space for the specified accuracy. In the following, we present a novel method to solve time dependent frequency equation using differential form of the equation, which then can be solved together with the dynamic equations of motion.

   Let us rewrite the clamped-mass frequency equation(8) as,

$$\begin{aligned} f(\beta_i, l) = {} & (1 + \cosh\beta_i \cos\beta_i) - \frac{M_L \beta_i}{\rho A l}(\cosh\beta_i \sin\beta_i - \sinh\beta_i \cos\beta_i) \\ & - \frac{J_L {\beta_i}^3}{\rho A l^3}(\cosh\beta_i \sin\beta_i + \sinh\beta_i \cos\beta_i) + \frac{M_L J_L {\beta_i}^4}{\rho^2 A^2 l^4}(1 - \cosh\beta_i \cos\beta_i) \quad = 0 \end{aligned} \tag{9}$$

   Since the frequency equation is continuous in β_i and the roots of the frequency equation are all distinct[18], we can differentiate equation(9) with respect to time,

$$\frac{df(\beta_i, l)}{dt} = \frac{\partial f(\beta_i, l)}{\partial \beta_i}\frac{d\beta_i}{dt} + \frac{\partial f(\beta_i, l)}{\partial l}\frac{dl}{dt} = 0 \tag{10}$$

   and rearrange to obtain,

$$\frac{d\beta_i}{dt} = \frac{f_1(\beta_i, l)}{f_2(\beta_i, l)} U(t) \tag{11}$$

   where

$$\begin{aligned} f_1(\beta_i, l) \quad = \quad & \frac{M_L}{\rho A l^2}\beta_i(\sinh\beta_i \cos\beta_i - \cosh\beta_i \sin\beta_i) \\ & - \frac{3J_L}{\rho A l^4}{\beta_i}^3(\sinh\beta_i \cos\beta_i + \cosh\beta_i \sin\beta_i) \\ & + \frac{4 M_L J_L}{\rho^2 A^2 l^5}{\beta_i}^4(1 - \cosh\beta_i \cos\beta_i) \end{aligned} \tag{12}$$

and

$$
\begin{aligned}
f_2(\beta_i, l) \quad = \quad & \left[1 + \frac{M_L}{\rho A l}\left(1 - \frac{J_L}{\rho A l^3}\beta_i{}^4\right)\right](\sinh\beta_i\cos\beta_i - \cosh\beta_i\sin\beta_i) \\
& - \frac{3J_L}{\rho A l^3}\beta_i{}^2(\sinh\beta_i\cos\beta_i + \cosh\beta_i\sin\beta_i) \\
& + \frac{2J_L}{\rho A l^3}\beta_i{}^3\left[\frac{2M_L}{\rho A l} - \left(1 + \frac{2M_L}{\rho A l}\right)\cosh\beta_i\cos\beta_i\right] \\
& - \frac{2M_L}{\rho A l}\beta_i\sinh\beta_i\sin\beta_i
\end{aligned} \tag{13}
$$

This ordinary differential equation(11) on β_i, is a function of the generalized variables (l and U) and can then be solved together with the dynamic equations of motion of the system, with the initial condition $\beta_i(t=0)$ solved from the frequency equation(9) for $l(t=0)$.

In summary, we can conclude that although the eigenfunctions $\psi_i(\eta)$ (see equation(6)) does not strictly satisfy the partial differential equation(3), the translating flexible beam can be quite accurately modeled as an instantaneous clamped-mass cantilever beam if the axial velocity U is constant and small compared to the group velocity U_g. We can assume that the separation of variables method can then be used, however, we need to take into account the slowly changing "eigen-frequencies". The time-dependent frequency equation due to the clamped-mass boundary conditions can be solved using a differential form of the equation together with the equations of motion.

3. DYNAMIC EQUATIONS OF MOTION

The dynamic equations of motion for the translating flexible beam shown in Figure 1 can be obtained by using the Lagrangian formulation[16]. Without loss of generality, let us suppose that the rotary inertia(J_L) at the tip of the translating beam is negligible. We assume that the elastic displacement $u(s,t)$ of a material point of the beam is discretized by assumed modes method in terms of the instantaneous clamped-mass eigen functions as,

$$
u(\tilde{\eta}, t) = \sum_{i=1}^{n} \psi_i(\tilde{\eta})\xi_i(t) \tag{14}
$$

where $\tilde{\eta} = \dfrac{s}{l(t)}$, $l(t)$ is the vibrating length of the translating beam, and n is the number of modes retained in the expansion. Then, we can write the velocity of the material point with reference to fixed coordinate system as,

$$\mathbf{v}(t) = \begin{pmatrix} U(t) \\ \sum_{i=1}^{n} \left[\psi_i(\tilde{\eta}) \frac{d\xi_i(t)}{dt} - \frac{\partial \psi_i(\tilde{\eta})}{\partial \tilde{\eta}} \xi_i(t) \frac{\tilde{\eta} U(t)}{l(t)} \right] \end{pmatrix} \tag{15}$$

Then the total kinetic energy of the system is given by,

$$T = \frac{1}{2}\rho A(l^0 - l(t))U^2(t) + \frac{1}{2}\rho A l(t) \int_0^1 \mathbf{v}(t)^T \mathbf{v}(t) d\tilde{\eta} + \frac{1}{2} M_L [\mathbf{v}(t)]_{\tilde{\eta}=1}^T [\mathbf{v}(t)]_{\tilde{\eta}=1} \tag{16}$$

and the total potential energy of the system (neglecting the axial and torsional vibrations of the translating beam) is given by[15],

$$V = \frac{1}{2} \frac{EI}{l^3(t)} \int_0^1 \left(\sum_{i=1}^{n} \frac{\partial^2 \psi_i(\tilde{\eta})}{\partial \tilde{\eta}^2} \xi_i(t) \right)^2 d\tilde{\eta} \tag{17}$$

The Lagrange's equations for the translating flexible beam with clamped boundary condition at the external input end for the generalized flexible variables, and for the rigid-body variable are given by,

for the rigid-body variable $l(t)$:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial U(t)} \right) - \frac{\partial T}{\partial l(t)} + \frac{\partial V}{\partial l(t)} = F \tag{18}$$

for the flexible deformation variable $\xi_i(t)$:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}_i(t)} \right) - \frac{\partial T}{\partial \xi_i(t)} + \frac{\partial V}{\partial \xi_i(t)} = 0 \tag{19}$$

Thus the equations of motion for the translating flexible beam utilizing the orthogonality of the eigen functions can be written in matrix form as,

$$\begin{pmatrix} m_{rr} & \mathbf{m}_{rf}^T \\ \mathbf{m}_{rf} & \mathbf{M}_{ff} \end{pmatrix} \begin{pmatrix} \ddot{l} \\ \ddot{\boldsymbol{\xi}} \end{pmatrix} + \begin{pmatrix} h_r \\ \mathbf{h}_f \end{pmatrix} + \begin{pmatrix} k_r \\ \mathbf{k}_f \end{pmatrix} = \begin{pmatrix} F \\ \mathbf{0} \end{pmatrix} \tag{20}$$

where F is the axial input force for the rigid-body axial motion of the beam as shown in Figure 1. Note that the subscripts r and f correspond to components of the rigid-body, and the flexible deformation variables respectively.

The elements of the positive-definite symmetric mass matrix $\mathbf{M}$ take on the expressions below,

$$\begin{aligned} m_{rr} &= \rho A l^0 + M_L + \frac{\rho A}{l(t)}\boldsymbol{\xi}(t)^T\mathbf{C}(t)\boldsymbol{\xi}(t) + \frac{M_L}{l^2(t)}\boldsymbol{\xi}(t)^T\widetilde{\mathbf{C}}(t)\boldsymbol{\xi}(t) && (21)\\ \mathbf{m}_{rf} &= -\rho A\mathbf{D}(t)\boldsymbol{\xi}(t) - \frac{M_L}{l(t)}\widetilde{\mathbf{D}}(t)\boldsymbol{\xi}(t) && (22)\\ \mathbf{M}_{ff} &= \rho A l(t)\mathbf{I} + M_L[\boldsymbol{\psi}(\tilde{\eta})]_{\tilde{\eta}=1}[\boldsymbol{\psi}(\tilde{\eta})]^T_{\tilde{\eta}=1} && (23) \end{aligned}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. The $n \times n$ matrices $\mathbf{C}$ and $\mathbf{D}$ are the domain integration of the eigen functions, whose ijth element is given by, $C_{ij} = \int_0^1 \tilde{\eta}^2 \frac{\partial \psi_i(\tilde{\eta})}{\partial \tilde{\eta}} \frac{\partial \psi_j(\tilde{\eta})}{\partial \tilde{\eta}} d\tilde{\eta}$, and $D_{ij} = \int_0^1 \tilde{\eta}\psi_i(\tilde{\eta}) \frac{\partial \psi_j(\tilde{\eta})}{\partial \tilde{\eta}} d\tilde{\eta}$ respectively. The ijth element of the $n \times n$ matrices $\widetilde{\mathbf{C}}$, and $\widetilde{\mathbf{D}}$ due to the end-mass boundary condition is given by, $\widetilde{C}_{ij} = \left[\frac{\partial \psi_i(\tilde{\eta})}{\partial \tilde{\eta}}\right]_{\tilde{\eta}=1} \left[\frac{\partial \psi_j(\tilde{\eta})}{\partial \tilde{\eta}}\right]_{\tilde{\eta}=1}$, and $\widetilde{D}_{ij} = [\psi_i(\tilde{\eta})]_{\tilde{\eta}=1} \left[\frac{\partial \psi_j(\tilde{\eta})}{\partial \tilde{\eta}}\right]_{\tilde{\eta}=1}$ respectively. Note that these domain dependent matrices are functions of the roots(β_i) of the frequency equation(9), and they will change with time(t) when the clamped-mass end conditions are used.

The terms h_r and $\mathbf{h}_f$ can be computed via the Christoffel symbols, i.e. via differentiation of the elements of the mass matrix, and are given by,

$$\begin{aligned} h_r &= -\frac{U^2(t)}{2l^2(t)}\boldsymbol{\xi}(t)^T\left[\rho A\mathbf{C}(t) + \frac{2M_L}{l(t)}\widetilde{\mathbf{C}}(t)\right]\boldsymbol{\xi}(t) + \frac{2U(t)}{l(t)}\boldsymbol{\xi}(t)^T\left[\rho A\mathbf{C}(t) + \frac{M_L}{l(t)}\widetilde{\mathbf{C}}(t)\right]\dot{\boldsymbol{\xi}}(t)\\ &\quad + \frac{U(t)}{2l(t)}\boldsymbol{\xi}(t)^T\left[\rho A\frac{d\mathbf{C}(t)}{dt} + \frac{M_L}{l(t)}\frac{d\widetilde{\mathbf{C}}(t)}{dt}\right]\boldsymbol{\xi}(t) - \dot{\boldsymbol{\xi}}(t)^T\left[\rho A\mathbf{D}(t) + \frac{M_L}{l(t)}\widetilde{\mathbf{D}}(t)\right]\dot{\boldsymbol{\xi}}(t)\\ &\quad - \frac{\rho A}{2}\dot{\boldsymbol{\xi}}(t)^T\dot{\boldsymbol{\xi}}(t) - \frac{M_L}{2U(t)}\dot{\boldsymbol{\xi}}(t)^T\frac{d}{dt}\left[\boldsymbol{\psi}(1)\boldsymbol{\psi}(1)^T\right]\dot{\boldsymbol{\xi}}(t) && (24)\\ \mathbf{h}_f &= -U(t)\left[\rho A\frac{d\mathbf{D}(t)}{dt} + \frac{M_L}{l(t)}\frac{d\widetilde{\mathbf{D}}(t)}{dt}\right]\boldsymbol{\xi}(t) - \frac{U^2(t)}{l(t)}\left[\rho A\mathbf{C}(t) + \frac{M_L}{l(t)}\widetilde{\mathbf{C}}(t)\right]\boldsymbol{\xi}(t)\\ &\quad - U(t)\left[\rho A\left(\mathbf{D}(t) - \mathbf{D}(t)^T\right) + \frac{M_L}{l(t)}\left(\widetilde{\mathbf{D}}(t) - \widetilde{\mathbf{D}}(t)^T\right)\right]\dot{\boldsymbol{\xi}}(t) + \rho A U(t)\dot{\boldsymbol{\xi}}(t)\\ &\quad + M_L\frac{d}{dt}\left[\boldsymbol{\psi}(1)\boldsymbol{\psi}(1)^T\right]\dot{\boldsymbol{\xi}}(t) && (25) \end{aligned}$$

It can be shown that a factorization of $\mathbf{h}(l, U, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}})$ exists

$$\mathbf{h}(l, U, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) = \begin{pmatrix} h_r \\ \mathbf{h}_f \end{pmatrix} = \begin{pmatrix} n_{rr} & \mathbf{n}_{rf} \\ \mathbf{n}_{fr} & \mathbf{N}_{ff} \end{pmatrix} \begin{pmatrix} U(t) \\ \dot{\boldsymbol{\xi}}(t) \end{pmatrix}$$

such that $\left[\frac{d}{dt}(\mathbf{M}) - 2\mathbf{N}\right]$ is skew-symmetric.

Note that the potential energy of the system (see equation(17)) is a function of the vibrating length of the translating beam $l(t)$, and hence using the Lagrange's equations(18-19), we get k_r and the stiffness vector $\mathbf{k}_f$ as,

$$k_r = -\frac{3EI}{2l^4(t)}\boldsymbol{\xi}(t)^T\mathbf{K}(t)\boldsymbol{\xi}(t) + \frac{EI}{2l^3(t)U(t)}\boldsymbol{\xi}(t)^T\frac{d\mathbf{K}(t)}{dt}\boldsymbol{\xi}(t) \tag{26}$$

$$\mathbf{k}_f = \frac{EI}{l^3(t)}\mathbf{K}(t)\boldsymbol{\xi}(t) \tag{27}$$

where the $n \times n$ diagonal stiffness coefficient matrix $\mathbf{K}$ is given by, $K_{ii} = \int_0^1 \left(\frac{\partial^2\psi_i(\tilde{\eta})}{\partial\tilde{\eta}^2}\right)^2 d\tilde{\eta}$. We note that the matrix $\mathbf{K}$ is time dependent, when the clamped-mass boundary conditions are used.

It should be remarked that the equations of motion(20-27) reduce to that of the bending vibration of a beam[16], when there is no translational motion (i.e. $\ddot{l} = U = 0$). It can also be observed that the rigid-body translational motion and flexural vibration motion are coupled, and it demonstrates the effects of elastic and translational motions on each other.

4. STABILITY ANALYSIS

In this section we discuss the stability characteristics of the free transverse vibratory motion of an axially moving flexible beam by Lyapunov's first method[19]. Let us rewrite the equations of motion of the system(20) in state-space form, defining the $(2n+2)$ dimensional state vector $\mathbf{x}(t) = \left(l(t), \boldsymbol{\xi}(t)^T, U(t), \dot{\boldsymbol{\xi}}(t)^T\right)^T$ as,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})F \tag{28}$$

where,

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} U(t) \\ \dot{\boldsymbol{\xi}}(t) \\ -h_{rr}(h_r + k_r) - \mathbf{h}_{rf}^T(\mathbf{h}_f + \mathbf{k}_f) \\ -\mathbf{h}_{rf}(h_r + k_r) - \mathbf{H}_{ff}(\mathbf{h}_f + \mathbf{k}_f) \end{pmatrix}$$

and

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} 0 \\ \mathbf{0} \\ h_{rr} \\ \mathbf{h}_{rf} \end{pmatrix}$$

with $\begin{pmatrix} h_{rr} & \mathbf{h}_{rf}^T \\ \mathbf{h}_{rf} & \mathbf{H}_{ff} \end{pmatrix} \triangleq \begin{pmatrix} m_{rr} & \mathbf{m}_{rf}^T \\ \mathbf{m}_{rf} & \mathbf{M}_{ff} \end{pmatrix}^{-1}$ and is given by

$$\mathbf{H} = \begin{pmatrix} (m_{rr} - \mathbf{m}_{rf}^T \mathbf{M}_{ff}^{-1} \mathbf{m}_{rf})^{-1} & -(m_{rr} - \mathbf{m}_{rf}^T \mathbf{M}_{ff}^{-1} \mathbf{m}_{rf})^{-1} \mathbf{m}_{rf}^T \mathbf{M}_{ff}^{-1} \\ -\mathbf{M}_{ff}^{-1} \mathbf{m}_{rf} (m_{rr} - \mathbf{m}_{rf}^T \mathbf{M}_{ff}^{-1} \mathbf{m}_{rf})^{-1} & (\mathbf{M}_{ff} - \mathbf{m}_{rf} m_{rr}^{-1} \mathbf{m}_{rf}^T)^{-1} \end{pmatrix}$$

Note that as the system mass matrix $\mathbf{M}$ is symmetric and positive definite, its inverse $\mathbf{H}$ exists, and is also symmetric and positive definite. By the fact that any principal submatrix of a positive definite matrix is positive definite, we note m_{rr}, $\mathbf{M}_{ff}$, h_{rr} and $\mathbf{H}_{ff}$ are also positive definite (and symmetric as $\mathbf{M}$ and $\mathbf{H}$ are symmetric).

As $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are continuously differentiable functions, we can linearize the non-linear system(28) around the reference state trajectory $\mathbf{x}_d(t) = (l_d(t), \mathbf{0}, U_d(t), \mathbf{0})^T$, to get,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}_d) + \mathbf{g}(\mathbf{x}_d) F_d(t) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}_d(t)} (\mathbf{x}(t) - \mathbf{x}_d(t)) + \mathbf{g}(\mathbf{x}_d)(F(t) - F_d(t)) + \mathbf{\Gamma} \tag{29}$$

where $\mathbf{\Gamma}$ represents higher-order terms in the Taylor's series expansion of $\mathbf{f}(\mathbf{x})$ about the reference state trajectory $\mathbf{x}_d(t)$. It should be noted that the linearized equations have meaning *only* if the higher-order terms in $(\mathbf{x}(t) - \mathbf{x}_d(t))$ are smaller than the linear terms, and indeed $\mathbf{\Gamma}$ remain small when $\mathbf{x}(t)$ remains *close* to $\mathbf{x}_d(t)$ in the sense that $\lim_{\|\mathbf{x}-\mathbf{x}_d\| \to 0} \frac{\| \mathbf{\Gamma} \|}{\| \mathbf{x} - \mathbf{x}_d \|} = 0$ where all norms are l_2-norms. We can then rewrite the linearized equations(29) in compact form by defining $\delta \mathbf{x}(t) \triangleq \mathbf{x}(t) - \mathbf{x}_d(t)$, and $\delta F(t) \triangleq F(t) - F_d(t)$ as,

$$\delta \dot{\mathbf{x}}(t) = \mathbf{A}(t) \delta \mathbf{x}(t) + \mathbf{B}(t) \delta F(t) \tag{30}$$

The time-varying state matrix $\mathbf{A}(t)$ takes the form,

$$\mathbf{A}(t) \triangleq \left[\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right]_{\mathbf{x}_d(t)} = \begin{pmatrix} 0 & \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ 0 & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}_p & \mathbf{0} & -\mathbf{A}_v \end{pmatrix} \tag{31}$$

where,

$$\mathbf{A}_p = \mathbf{H}_{ff} \left[\frac{EI}{l_d^3(t)} \mathbf{K} - U_d(t)(\rho A \frac{d\mathbf{D}}{dt} + \frac{M_L}{l_d(t)} \frac{d\widetilde{\mathbf{D}}}{dt}) - \frac{U_d^2(t)}{l_d(t)}(\rho A \mathbf{C} + \frac{M_L}{l_d(t)} \widetilde{\mathbf{C}})\right] \tag{32}$$

$$\mathbf{A}_v = \mathbf{H}_{ff} \left[\rho A U_d(t) \mathbf{I} + M_L \frac{d}{dt}[\boldsymbol{\psi}(1) \boldsymbol{\psi}(1)^T] + U_d(t)(\rho A [\mathbf{D}^T - \mathbf{D}] + \frac{M_L}{l_d(t)} [\widetilde{\mathbf{D}}^T - \widetilde{\mathbf{D}}])\right] \tag{33}$$

The control distribution matrix $\mathbf{B}(t)$ takes the form,

$$\mathbf{B}(t) \triangleq \mathbf{g}(\mathbf{x}_d(t)) = \begin{pmatrix} 0 \\ \mathbf{0} \\ h_{rr} \\ \mathbf{0} \end{pmatrix} \tag{34}$$

Lemma. *The system of equations governing the transverse vibratory motion of an axially moving beam is stable while the beam undergoes uniform extension and is unstable while the beam undergoes uniform retraction.*

PROOF. As the rigid-body and the vibratory motion are decoupled in the linearized form, we write only the equations governing the vibratory motion from equation(30) as,

$$\ddot{\boldsymbol{\xi}}(t) + \mathbf{A}_v\dot{\boldsymbol{\xi}}(t) + \mathbf{A}_p\boldsymbol{\xi}(t) = \mathbf{0} \tag{35}$$

and observe that the necessary condition for $\boldsymbol{\xi}(t)$ to be stable is that the matrix $\mathbf{A}_v$ is positive definite. From the equation(23) and using the fact that the roots of the frequency equation is a function of the vibrating length of the beam, we can rewrite $\mathbf{A}_v$ as,

$$\begin{aligned} \mathbf{A}_v &= \mathbf{H}_{ff}\left[\frac{\partial}{\partial l}[\mathbf{M}_{ff}] + (\rho A[\mathbf{D}^T - \mathbf{D}] + \frac{M_L}{l_d(t)}[\widetilde{\mathbf{D}}^T - \widetilde{\mathbf{D}}])\right] U_d(t) \\ &= \mathbf{H}_{ff}\left[\mathbf{G}_s + \mathbf{G}_{ss}\right] U_d(t) \end{aligned}$$

Note that the matrix $\mathbf{G}_{ss}$ is skew-symmetric, and the matrix $\mathbf{G}_s$ is positive definite from the property of the system mass matrix. Hence the term inside parentheses is a positive definite matrix. As noted earlier $\mathbf{H}_{ff}$ is positive definite, and hence $\mathbf{A}_v$ is positive definite only if $U_d(t)$ remains positive. $U_d(t)$ is positive while the beam undergoes uniform axial extension which implies that the eigenvalues of the system(35) will have negative real parts during extension of the beam, giving stable dynamic response. On the other hand, as $U_d(t)$ is negative during the uniform retraction of the beam, $\mathbf{A}_v$ is negative definite giving rise to unstable dynamic response. It should be mentioned here that although we have examined the stability status of the reference state trajectory for the linearized equations, *locally* the stability results hold true for the nonlinear system(28) also (see pp.188-190 of [19]). ■

It should be mentioned that during extension of the flexible link, the amplitude of modal position variables increases as the frequency of vibration decreases. However the motion is stable, since eigenvalues of the system(35) have negative real parts. This is observed in our numerical simulations (see section 5). On the other hand, during retraction of the link, the frequency increases and the amplitude of modal position variables decreases. However

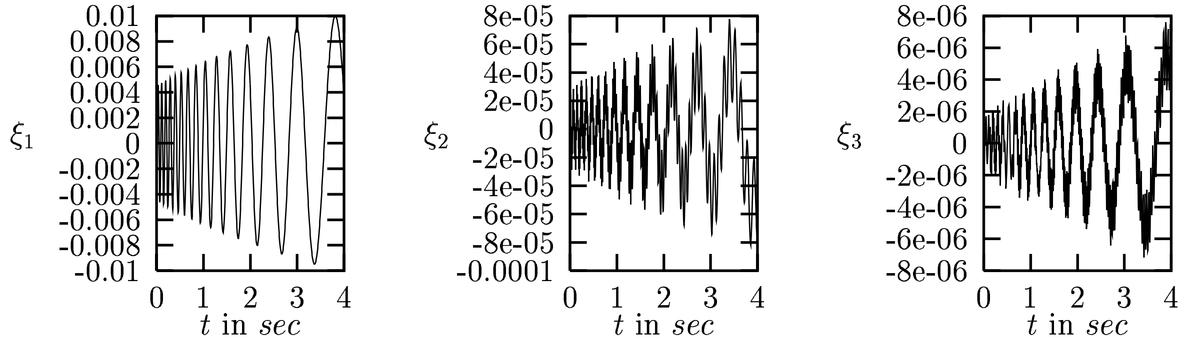

Figure 2: Position response of flexible modal variables for the uniform extension of the beam ($l_d(t) = 0.5 + U_d t, U_d = 0.5m/s$).

the motion is unstable as the eigenvalues of the system have positive real parts. Again this is observed in our numerical simulations. The above results are in contrast to those reported by Wang and Wei[3], nonetheless, they are in agreement with the recently reported results of Stylianou and Tabarrok[6].

5. NUMERICAL RESULTS

We now present some numerical results for a typical Euler-Bernoulli beam undergoing both rigid-body axial and flexural vibratory motions. The beam is assumed to be uniform with parameter values $\rho A = 0.5kg/m$, $EI = 100Nm^2$, $M_L = 0.125kg$, $l^0 = 3.0m$, and $U_g = 4.714m/s$. In this section, we study only the induced vibration of the beam for prescribed *a priori* rigid-body uniform axial motion. We present results for two cases of uniform axial motion[2]: one, the beam extending with a constant axial velocity of 0.5m/s, and the other, the beam retracting with a constant axial velocity of −0.5m/s. The first-order differential equations corresponding to the transverse vibratory motion (see equation(28)) for the specified rigid-body variables $(l_d(t), U_d(t))$ are solved by the variable step, variable order (of interpolation), predictor-corrector (PECE), Adams algorithm[20] on a SUN-SPARC 10 Workstation. The initial conditions for flexible modal position variables are so chosen that the initial potential energy of the system(equation(17)) is same for simulation of both the cases of axial motion.

Figure 2 shows the response of flexible modal position variables during the uniform

[2] In this simulation, $U/U_g \approx 0.11$, and the results show that the assumed modes model give reasonably accurate results even for such a large U/U_g ratio.

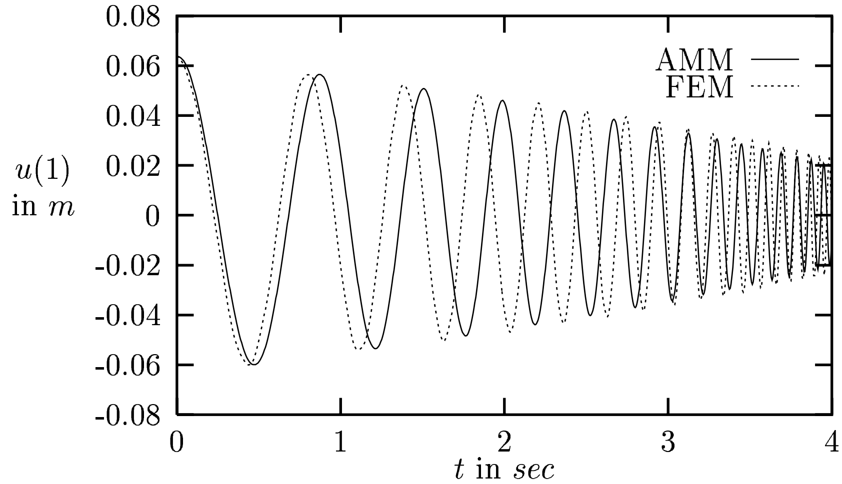

Figure 3: Comparison of tip deflection for the uniform retraction of the beam.

extension of the beam. We observe that the contribution from higher modes to the tip deflection of the beam is several orders smaller compared to the first mode. This indicates the convergence of assumed modes model for the problem at hand. In Figure 3, we compare the tip deflection during uniform retraction of the beam obtained using the instantaneous mode shapes with those obtained from a FEM based model, and it can be seen that they match quite accurately. Figure 3 also shows that the vibratory response of the beam based on FEM has a higher fundamental natural frequency than the assumed modes model as reported earlier[11]. Furthermore, although the results are not shown here, it has been observed that the rate of work performed by the axial force (computed from the first equation of the equations of motion(20)) is equal to the computed rate of change of the total energy of the system. Thus, this study demonstrates the validity of the numerical simulation, and the feasibility of assumed modes model for accurate prediction of vibratory behavior during uniform axial motions of the flexible beam.

To illustrate the stable response of modal variables during extension vis-a-vis unstable response during retraction of the beam, we have plotted the position and velocity response of mode-1 variable in Figures 4 and 5. We observe that the amplitude of mode-1 position variable increases during the uniform extension case, and its velocity component decreases, as opposed to the increase in magnitude of the velocity of mode-1 variable during uniform retraction of the beam. It should be mentioned here that the increase in amplitude of modal position variables during extension and the decrease during retraction is as anticipated,

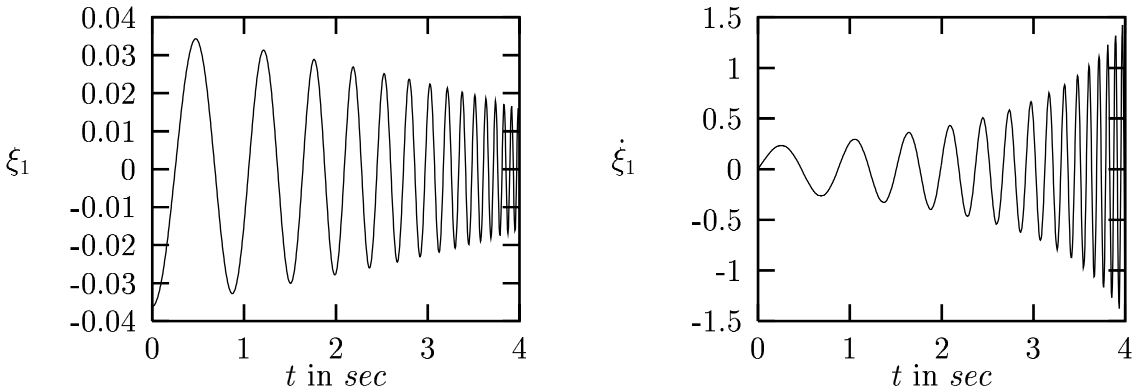

Figure 4: Position and velocity response of flexible modal variables for the uniform retraction of the beam ($l_d(t) = 2.5 + U_d t, U_d = -0.5 m/s$).

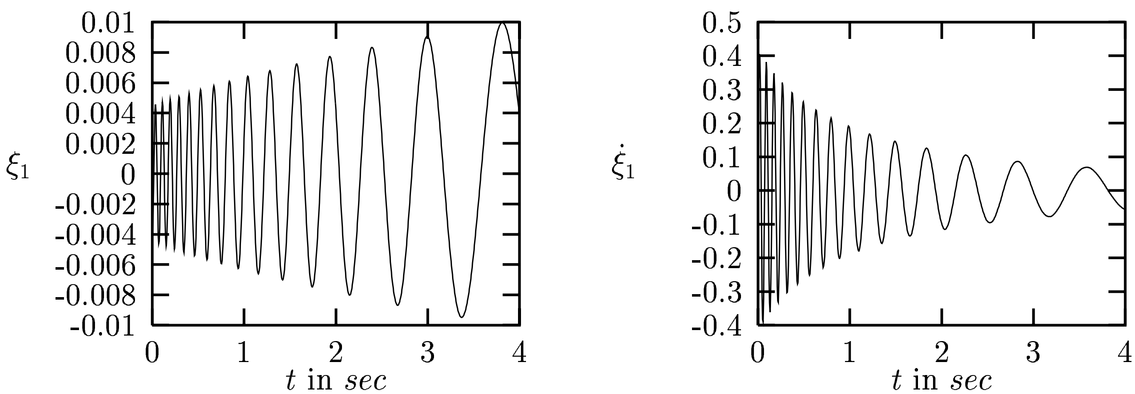

Figure 5: Position and velocity response of flexible modal variables for the uniform extension of the beam ($l_d(t) = 0.5 + U_d t, U_d = 0.5 m/s$).

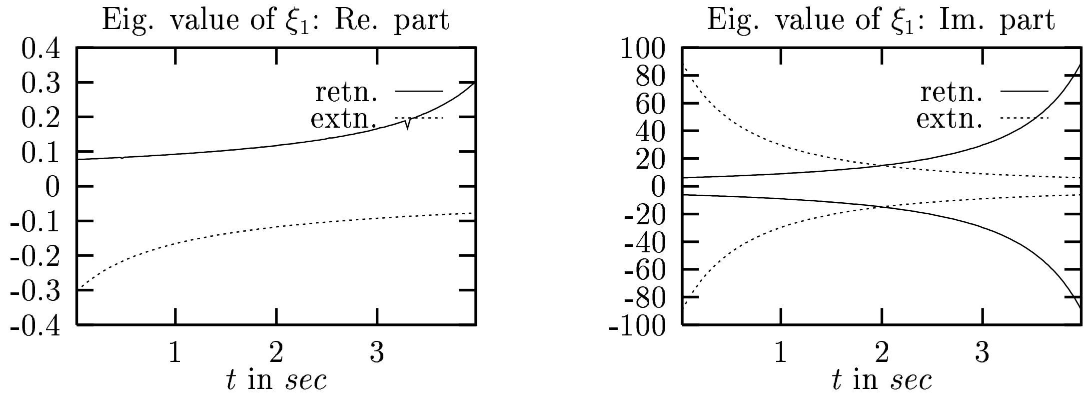

Figure 6: Time history of real and imaginary parts of the mode-1 eigenvalues for the uniform retraction and extension of the beam.

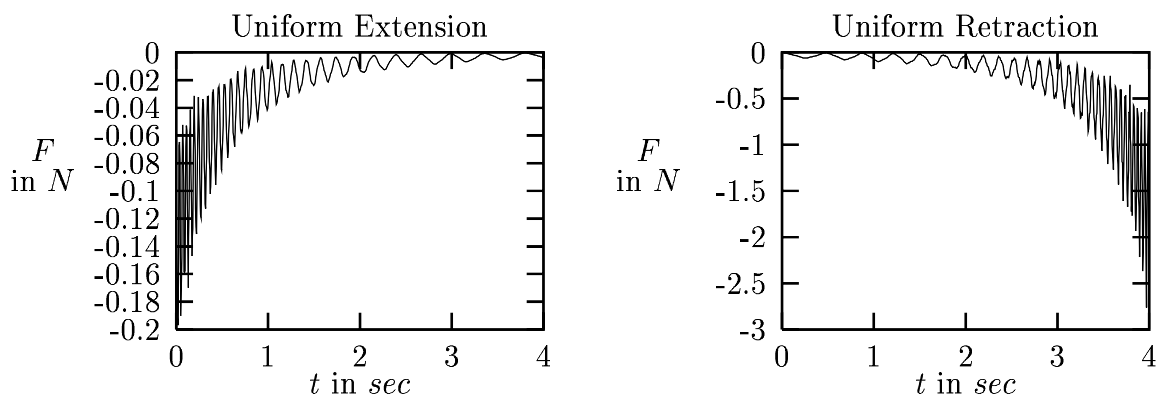

Figure 7: Time history of the calculated force F during uniform axial extension and retraction of the beam.

because the flexible beam becomes "softer" as the length increases and "stiffer" as the length reduces[5]. This is also evident by observing the imaginary part of eigenvalues (which correspond to the natural frequency of modal variables) in Figure 6. We note that the real part of eigenvalues corresponding to modal variables remain positive during retraction of the beam, as opposed to the negative real parts of eigenvalues during the extension of the beam. In Figure 7, we have plotted the force F computed from the equation of motion in the axial direction (see the first equation in (20)) for the specified time histories of $l_d(t)$, $U_d(t)$ and the computed time histories of $\boldsymbol{\xi}(t)$, $\dot{\boldsymbol{\xi}}(t)$ (from the second equation in (20)). It is interesting to observe that this force remains negative (i.e. pointing in the negative Z

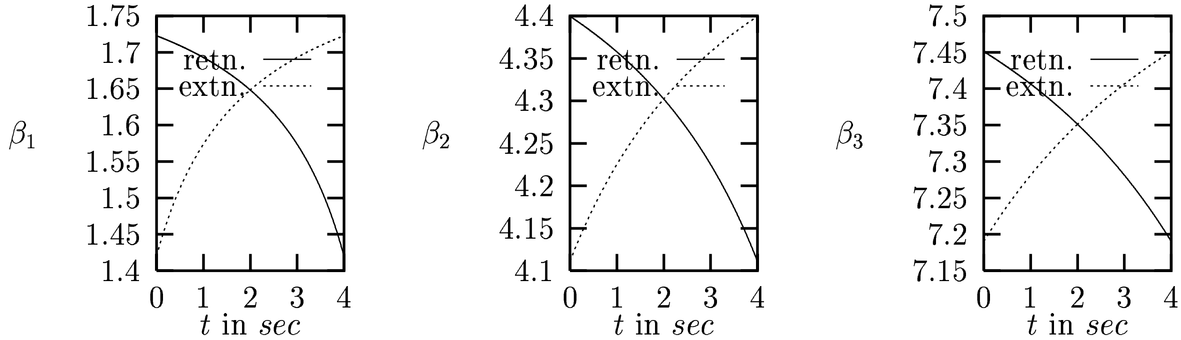

Figure 8: Time history of the roots(β_i) of clamped-mass frequency equation.

direction, see Figure 1) during both uniform extension and uniform retraction of the beam. This fact is intuitively clear – to counter the centrifugal force due to transverse vibration and to maintain constant axial velocity, force has to be applied in the negative Z direction. However, we have not been able to prove this analytically from the equations of motion (equation(20)). We also note that the force becomes highly oscillatory when length of the beam reduces. This is mainly due to the increase in the frequency of oscillation of flexible modal variables when vibrating length of the beam reduces during axial motion. Finally, we show in Figure 8, the continuous time dependency of roots(β_i) of the frequency equation(9) due to the end-mass boundary condition for both uniform retraction and extension of the beam.

6. SUMMARY

In this paper, we have presented a discussion on the applicability of using separation of variables and the assumed modes method for discretizing a translating flexible beam. Based on the concept of group velocity of the dispersive waves in beam, we presented a non-dimensionalized Euler-Bernoulli equation for the translating beam. We showed that if the beam is translating at a constant, slow (compared to the group velocity) speed, the principle of separation of variables can be applied. We showed that when the mass and rotary inertia of the load are comparable to that of the flexible beam, the mass end-conditions are more accurate to use for the choice of proper eigen-functions. The clamped-mass boundary conditions, however, lead to a time-dependent frequency equation. We presented a novel method to solve this time-dependent frequency equation by using a differential form of the frequency equation. We then derived the closed form equations of motion for a translating flexible beam using the Lagrangian formulation of dynamics in conjunction with the assumed modes method. We showed using the Lyapunov's first method that the dynamic response of flexible modal variables become unstable during retraction of a flexible link, compared to the stable dynamic response during extension of the link. The above results were illustrated with numerical simulations for a typical beam undergoing uniform axial motion.

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REFERENCES

1. B. TABARROK, C. M. LEECH, and Y. I. KIM 1974 *Journal of Franklin Institute* **297**(3), 201–220. On the dynamics of axially moving beams.

2. K. W. BUFFINTON and T. R. KANE 1985 *International Journal of Solids Structure* **21**(7), 617–643. Dynamics of beam moving over supports.

3. P. K. C. WANG and J. D. WEI 1987 *Journal of Sound and Vibration* **116**(1), 149–160. Vibrations in a moving flexible robot arm.

4. J. YUH and T. YOUNG 1991 *Transactions of the ASME Journal of Dynamic Systems, Measurement, and Control* **113**, 34–40. Dynamic modeling of an axially moving beam in rotation: simulation and experiment.

5. S. S. K. TADIKONDA and H. BARUH 1992 *Transactions of the ASME Journal of Dynamic Systems, Measurement, and Control* **114**, 422–427. Dynamics and control of a translating flexible beam with a prismatic joint.

6. M. STYLIANOU and B. TABARROK 1994 *Journal of Sound and Vibration* **178**(4), 433–481. Finite element analysis of an axially moving beam, Part I: Time integration; Part II: Stability analysis.

7. A. K. MISRA and V. J. MODI 1982 *Journal of Guidance* **5**(3), 278–285. Deployment and retrieval of shuttle supported tethered satellites.

8. K. TSUCHIYA 1983 *Journal of Guidance* **6**(1), 100–103. Dynamics of a spacecraft during extension of flexible appendages.

9. C. W. WAMPLER, K. W. BUFFINTON, and S. H. JIA 1985 *Transactions of the ASME Journal of Applied Mechanics* **152**, 465–470. Formulation of equations of motion for systems subject to constraints.

10. M. S. JANKOVIC 1983 *Journal of Guidance* **7**(1), 128. Comments on Dynamics of a spacecraft during extension of flexible appendages.

11. R. J. THEODORE and A. GHOSAL 1995 *The International Journal of Robotics Research* **14**(2), 91–111. Comparison of the assumed modes and finite element models for flexible multi-link manipulators.

12. R. ELMARAGHY and B. TABARROK 1975 *Journal of Franklin Institute* **300**(1), 25–39. On the dynamic stability of an axially oscillating beam.

13. J. ZAJACZKOWSKI and J. LIPINSKI 1979 *Journal of Sound and Vibration* **63**(1), 9–18. Instability of the motion of beam of periodically varying length.

14. J. ZAJACZKOWSKI and G. YAMADA 1980 *Journal of Sound and Vibration* **68**(2), 173–180. Further results on instability of the motion of a beam of periodically varying length.

15. S. TIMOSHENKO, D. H. YOUNG, and W. WEAVER Jr. 1974 *Vibration Problems in Engineering.* New York: John Wiley; fourth edition.

16. L. MEIROVITCH 1986 *Elements of Vibration Analysis.* New York: McGraw-Hill Book Company; second edition.

17. G. B. WHITHAM 1974 *Linear and Nonlinear Waves.* New York: John Wiley.

18. P. A. A. LAURA, J. L. POMBO, and E. A. SUSEMIHL 1974 *Journal of Sound and Vibration* **37**(2), 161–168. A note on the vibrations of a clamped-free beam with a mass at the free end.

19. M. VIDYASAGAR 1978 *Nonlinear Systems Analysis.* Englewood Cliffs, New Jersey: Prentice-Hall.

20. L. F. SHAMPINE and M. K. GORDON 1975 *Computer Solution of Ordinary Differential Equations: The Initial Value Problem.* San Francisco: W. H. Freeman.