A differential geometric method for kinematic analysis of two- and three-degree-of-freedom rigid body motions

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Abstract

In this paper, we present a novel differential geometric characterization of two- and three-degree-of-freedom rigid body kinematics using a metric defined on dual vectors. The instantaneous angular and linear velocities of a rigid body are expressed as a dual velocity vector, and dual inner product is defined on this dual vector resulting in a positive semi-definite and symmetric dual matrix. We show that the maximum and minimum magnitude of the dual velocity vector, for a unit speed motion, can be obtained as eigenvalues of this dual matrix. Furthermore, we show that the tip of the dual velocity vector lies on a dual ellipse for a two-degree-of-freedom motion and on a dual ellipsoid for a three-degree-of-freedom motion. In this manner velocity distribution of a rigid body can be studied algebraically in terms of the eigenvalues of a dual matrix or geometrically with the dual ellipse and ellipsoid. The second-order properties of the two- and three-degree-of-freedom motions of a rigid body are also obtained from the derivatives of the elements of the dual matrix. This results in a definition of the geodesic motion of a rigid body. The theoretical results are illustrated with the help of a spatial 2R and a parallel three-degree-of-freedom manipulator.

1 Introduction

The instantaneous kinematics of multi-degree-of-freedom rigid body motion is important in several areas, including mechanism analysis and synthesis, robot control and motion planning, and spacecraft guidance and control. There are many approaches for studying instantaneous spatial kinematics of rigid body motions. A common way is to analyze the trajectories of points and lines which are the elements of the rigid body(see, for example, [1] through [7]). Another approach is to use the rotation matrix and a translation vector and obtain instantaneous invariants(see, for example, [8] through [11]) and a third method is to use

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screws, twists and wrenches(see, for example, [12] through [15]), and yet another approach is to use the notion of kinematic mappings[16].

In this paper, we take a re-look at instantaneous two- and three-degree-of-freedom rigid body kinematics using classical differential-geometric tools and techniques. These techniques are for studying differential properties of curves and surfaces and have therefore only been used, in kinematics, to study trajectories of points (also lines and planes)in rigid body motions. They do not directly apply to differential kinematic analysis of rigid body motions and often other approaches are usually used to evaluate velocity and acceleration distributions of rigid body motions. In this paper we extend these differential geometric tools to the study of rigid body motions rather than their point or line trajectories. We obtain such an extension through the definition and use of a metric in the space of dual numbers and vectors[1]. We represent the linear and angular velocity of a rigid body moving in three dimensional space, $\Re^3$, as a dual vector. We define a metric based on the definition of an inner product of two dual vectors as a *dual number*[17, 18]. This allows us to obtain the maximum and minimum of the dual velocity vector as eigenvalues of a *positive semi-definite and symmetric* dual matrix associated with the metric. We show that for a unit speed motion, the tip of the dual velocity vector lies on a dual ellipse and a dual ellipsoid for two- and three-degree-of-freedom motions respectively.

Our approach for differential kinematic analysis is analogous to study of the second fundamental form of a surface and the concepts of Christoffel symbols, Gaussian and geodesic curvatures. Similar to the notion of a geodesic on a surface, we define a *geodesic motion* of a rigid body where the components of the dual tangential acceleration are zero. For a general 2R spatial manipulator, all the dual Christoffel symbols have real parts as zero. In addition, the cylindroid and its principle pitches are same everywhere for a 2R spatial manipulator since the first-order properties are independent of the joint variables. The motion of the central cylindroid axis is a two parameter family of lines obtained by translation and rotation, and the envelope of this family is a 'quadratic cone'. The geodesics are straight lines in the configuration space and these map to a particular family of lines. The foot of the perpendicular from the origin to the line along the central axis of the cylindroid lies on a curve on the 'quadratic cone'. It is also shown that the 'geodesic curve' can be closed or open depending on initial conditions. For the three-degree-of-freedom parallel manipulator the pitches associated with the line congruence are different at different configurations and the dual Christoffel symbols are not all zero.

[1]Dual numbers, first introduced by Clifford[19], have been used extensively in kinematics(see, for example, [20] through [23]).

We hope that this new approach to differential kinematic analysis of rigid body motions can lead to better understanding of velocity and acceleration distributions in robot manipulators and other complex motions. The approach can also lead to other new theoretical results such as the notion of geodesic motions described in this paper.

The paper is organized as follows: In section 2, we present, in brief, the definition of an inner product of two dual vectors as a dual number, the representation of lines and screws in terms of dual vectors, and the eigenvalues and eigenvectors of a dual matrix. In section 3, we consider differential kinematics of two and three degree-of-freedom motion of a rigid body in $\Re^3$ and show that the maximum and minimum dual velocities can be obtained as eigenvalues of a positive semi-definite, symmetric dual matrix. Furthermore, we show that the tip of the dual velocity vector lies on a dual ellipse or ellipsoid. We also obtain second-order properties and obtain expressions for the dual Christoffel symbols. In section 4, we illustrate the theoretical results with the help of a spatial 2R serial manipulator and three-degree-of-freedom parallel manipulator. In section 5, we summarize the main results of this paper.

2 Mathematical preliminaries

A dual number, $\hat{a}$, has the form $a + \epsilon a_0$ where a and a_0 are real numbers and $\epsilon^2 = \epsilon^3 ... = 0$. A dual vector, $\hat{\mathbf{A}}$, has the form $\mathbf{a} + \epsilon \mathbf{a}_0$ where $\mathbf{a}$ and $\mathbf{a}_0$ are real vectors in $\Re^3$. The inner product of two dual vectors $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ can be defined as[17]

$$< \hat{\mathbf{A}}, \hat{\mathbf{B}} >= \mathbf{a} \cdot \mathbf{b} + \epsilon(\mathbf{a} \cdot \mathbf{b}_0 + \mathbf{b} \cdot \mathbf{a}_0) \quad (1)$$

It may be noted that the above inner product is invariant to the choice of the origin of the coordinate system used to describe $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$. The inner product is also different from the inner product defined in [24](or see the English translation [25]) where the inner product has been defined as $(\mathbf{a} \cdot \mathbf{b}_0 + \mathbf{b} \cdot \mathbf{a}_0)$.

A line in $\Re^3$ can be described as a dual vector as

$$\hat{\mathcal{L}} = \mathbf{Q} + \epsilon \mathbf{Q}_0 \quad (2)$$

where $\mathbf{Q}$ denotes the direction of the line, and $\mathbf{Q}_0 = \mathbf{r} \times \mathbf{Q}$ is the moment of the line with $\mathbf{r}$ as the position vector of any point on the line from an origin. There are 4 independent parameters in $\mathbf{Q}$ and $\mathbf{Q}_0$ since $|\mathbf{Q}| = 1$ and $\mathbf{Q} \cdot \mathbf{Q}_0 = 0$. The foot of the perpendicular from the origin to the line, denoted by, $\mathbf{r}_0 = (x, y, z)^T$, is given as

$$\mathbf{r}_0 = \mathbf{Q} \times \mathbf{Q}_0 \quad (3)$$

The inner product of two lines follows from equation (1) and we have

$$\begin{aligned} < \hat{\mathcal{L}}_1, \hat{\mathcal{L}}_2 > &= \mathbf{Q}_1 \cdot \mathbf{Q}_2 + \epsilon(\mathbf{Q}_1 \cdot \mathbf{Q}_{02} + \mathbf{Q}_2 \cdot \mathbf{Q}_{01}) \\ &= \cos\phi - \epsilon d \sin\phi \end{aligned} \tag{4}$$

where ϕ and d are the angle and the shortest distance respectively between the two lines.

A screw can be also described as a dual vector as

$$\hat{\mathcal{S}} = \mathbf{S} + \epsilon \mathbf{S}_0 \tag{5}$$

where, in terms of line coordinates, $\mathbf{S} = \mathbf{Q}$ and $\mathbf{S}_0 = \mathbf{Q}_0 + h\mathbf{Q}$. In the previous equation h is called the pitch of the screw and is the ratio of the translational displacement to the rotational displacement. A screw has 5 independent parameters, i.e., 4 associated with the line along the screw and a pitch.

The inner product of two screws follows from equations (1) and (4), and we have

$$\begin{aligned} < \hat{\mathcal{S}}_1, \hat{\mathcal{S}}_2 > &= \mathbf{S}_1 \cdot \mathbf{S}_2 + \epsilon(\mathbf{S}_1 \cdot \mathbf{S}_{02} + \mathbf{S}_2 \cdot \mathbf{S}_{01}) \\ &= \cos\phi + \epsilon((h_1 + h_2)\cos\phi - d\sin\phi) \end{aligned} \tag{6}$$

where h_1 and h_2 are the pitches associated with the two screws. The inner product of a screw with itself, from equation (6), is $1 + \epsilon(2h_i)$.

The angular velocity, $\boldsymbol{\omega}$, and the linear velocity, $\mathbf{v}$, of a point on a rigid body, can be together considered as a dual vector of the form

$$\hat{\mathcal{V}} = \boldsymbol{\omega} + \epsilon \mathbf{v}_p \tag{7}$$

The quantity $\hat{\mathcal{V}}$ have also been called a twist and a motor, and can be thought of as a screw together with a magnitude. In terms of line coordinates, $\hat{\mathcal{V}}$ is given as

$$\hat{\mathcal{V}} = |\boldsymbol{\omega}|(\mathbf{Q} + \epsilon(\mathbf{Q}_0 + h\mathbf{Q})) \tag{8}$$

where $|\boldsymbol{\omega}|$ and $\mathbf{Q}$ give the magnitude and the direction of the angular velocity vector respectively. The pitch of the screw may be obtained as

$$h = \frac{\boldsymbol{\omega} \cdot \mathbf{v}_p}{|\boldsymbol{\omega}|^2} \tag{9}$$

The eigenvalues of a $n \times n$ dual matrix $[\hat{\mathbf{g}}]$ can be obtained from

$$[\hat{g}]\mathbf{x} - \hat{\lambda}\mathbf{x} = \mathbf{0} \tag{10}$$

where $\hat{\lambda}$ is an eigenvalue of the matrix $[\hat{g}]$, and $\mathbf{x}$ is the corresponding eigenvector. The eigenvalue is a dual number of the form $\hat{\lambda} = \lambda + \epsilon\lambda_0$, where λ and λ_0 are the real and dual parts of $\hat{\lambda}$ respectively. Similarly, the matrix $[\hat{g}]$ may be written in terms of the real, and dual components as $[\hat{g}] = [g] + \epsilon[g_0]$.

The dual characteristic polynomial of $[\hat{g}]$ is obtained from $\det([\hat{g}] - \hat{\lambda}[I]) = 0$, where $[I]$ is a real $n \times n$ identity matrix. Using the real and dual components of $[\hat{g}]$ and $\hat{\lambda}$, the determinant can be expanded as $\det[([g] - \lambda[I]) + \epsilon([g_0] - \lambda_0[I])]$. The determinant can be further expanded in terms of the column vectors of the matrices $([g] - \lambda[I])$ and $([g_0] - \lambda_0[I])$ as

$$\begin{aligned} &\det([g_1|g_2|....|g_n]) \\ &+ \epsilon(\det([g_{01}|g_2|....|g_n]) + \det([g_1|g_{02}|....|g_n]) + ... + \det([g_1|g_2|....|g_{0n}])) = 0 \end{aligned} \tag{11}$$

where g_i and g_{0i} represent the i^{th} column of $([g]-\lambda[I])$ and $([g_0]-\lambda_0[I])$ respectively. Equation (11) can be expanded to yield the characteristic polynomial of $[\hat{g}]$ in the form

$$\sum_{r=0}^{n} (a_r + \epsilon a_{r0})(\lambda^r + \epsilon r \lambda^{r-1}\lambda_0) = 0 \tag{12}$$

where a_r and a_{r0} are the coefficients of the dual characteristic polynomial. Equating the real and dual parts of the above equation to zero separately, we get polynomial equations of the form

$$\begin{aligned} \sum_{r=0}^{n} a_r \lambda^r &= 0 \\ \sum_{r=0}^{n} a_r r \lambda^{r-1}\lambda_0 + a_{r0}\lambda^r &= 0 \end{aligned} \tag{13}$$

Solving the first of the above pair of equations we can get n values of λ, and on substituting λ in the second equation we can *linearly* solve for the n λ_0's as

$$\lambda_0 = -\frac{\sum_{r=0}^{n} a_{r0}\lambda^r}{\sum_{r=0}^{n} a_r r \lambda^{r-1}} \qquad (\lambda \neq 0) \tag{14}$$

If $\lambda = 0$, by applying L'Hospitals rule, we can easily show that λ_0 is also 0.
For $\lambda \neq 0$, a *principal pitch*, h^*, can be derived as

$$h^* = \frac{\lambda_0}{2\ \lambda} \tag{15}$$

When λ and λ_0 are zero, the principal pitch is not defined. In such a situation, the rigid body undergoes a *pure translation* and, in accordance with classical screw theory, we can state that the pitch is infinite. It may be noted that λ_0 maybe zero for non-zero λ. In such a case h^* is zero, and, in accordance with classical screw theory, the rigid body undergoes a *pure rotation.*

The eigenvectors, $\mathbf{x}$, may be computed from the standard eigen-problem corresponding to the real part, namely,

$$\begin{aligned} ([g] - \lambda[I])\mathbf{x} &= 0 \\ |\mathbf{x}| &= 1 \end{aligned} \tag{16}$$

3 Two- and three-degree-of-freedom motion of a rigid body

Before we discuss the motion of a rigid body, we briefly present the concept of a velocity ellipse and ellipsoid[5] traced by an arbitrary point on a moving rigid body. Our approach is to extend the differential-geometric method and results of point trajectories to the instantaneous kinematics of the entire rigid body by use of dual numbers and vectors.

3.1 Point trajectory

Consider a rigid body B undergoing a multi-degree-of-freedom motion, with respect to a fixed coordinate system. The trajectory traced by an arbitrary point in B, with Cartesian coordinates (x, y, z), can be expressed as a set of equations in terms of the n independent motion parameters θ_i, $i = 1, 2, ..., n$ [2] as[3]

$$(x, y, z)^T = \boldsymbol{\psi}(\theta_1, ..., \theta_n) \tag{17}$$

The velocity at any point, $\mathbf{p}$, on the point trajectory can be written as

$$\mathbf{v} = \sum_{i=1}^{n} \boldsymbol{\psi}_i \dot{\theta}_i = [J(\boldsymbol{\psi})]_{\mathbf{p}} \dot{\boldsymbol{\theta}} \tag{18}$$

where $\dot{\theta}_i$ is the time derivative of θ_i, $\boldsymbol{\psi}_i$ is $\partial\boldsymbol{\psi}/\partial\theta_i$ evaluated at $\mathbf{p}$, $\dot{\boldsymbol{\theta}}$ is the vector $(\dot{\theta}_1, ..., \dot{\theta}_n)^T$ and $[J(\boldsymbol{\psi})]_{\mathbf{p}}$ is the Jacobian matrix of $\boldsymbol{\psi}$ evaluated at $\mathbf{p}$. By varying $\dot{\boldsymbol{\theta}}$, we can get any arbitrary velocity $\mathbf{v}$ at $\mathbf{p}$. It is more instructive to look at the variation of $\mathbf{v}$ with a normalizing

[2] For $n = 1, 2, 3$ the point trajectory is a curve, a surface and a solid region in $\Re^3$ respectively. For $n > 3$ we have a redundant motion.

[3] These are known as the forward kinematics equations in manipulator kinematics.

constraint of the form $\dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} = k^2$. For $k = 1$, we have a *unit speed motion* and by varying k one can get all possible velocities, $\mathbf{v}$, at the point $\mathbf{p}$ under consideration.

The dot product of the velocity with itself can be written as

$$\mathbf{v} \cdot \mathbf{v} = \dot{\boldsymbol{\theta}}^T [g] \dot{\boldsymbol{\theta}} \tag{19}$$

where the matrix elements, g_{ij}, are the dot products $(\boldsymbol{\psi}_i \cdot \boldsymbol{\psi}_j), i, j = 1, 2, .., n$. The matrix $[g]$ is *symmetric* and *positive definite* and its elements(in the language of differential geometry) define a *metric* in the tangent space[26]. We make the following observations from the definition of $[g]$ and equations (18) and (19):

- If $[g]$ is non-singular(i.e, $\det[g] \neq 0$), then we can write

$$\mathbf{v}^T ([J][g]^{-1})([J][g]^{-1})^T \mathbf{v} = \dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} \tag{20}$$

  The matrix $([J][g]^{-1})([J][g]^{-1})^T$ is symmetric and positive definite, and for $\dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} = 1$, the tip of the velocity vector $\mathbf{v}$ lies on an ellipsoid. For two-degree-of-freedom motions the tip of the vector $\mathbf{v}$ lies on an ellipse in the tangent plane as shown schematically in figure 1. In classical differential geometry of surfaces this defines the so-called *Tissot's indicatrix*(see pages 145-155 of Strubecker[27]).

- The principal values of $\mathbf{v}^2$ subject to constraint $\dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} = 1$ can be obtained by solving the eigenvalue problem

$$[g]\dot{\boldsymbol{\theta}} - \lambda \dot{\boldsymbol{\theta}} = 0 \tag{21}$$

  The maximum, minimum and intermediate values of $|\mathbf{v}|$ are the square roots of the maximum, minimum, and intermediate eigenvalues of $[g]$, namely $\sqrt{\lambda_{max}}$, $\sqrt{\lambda_{min}}$, and $\sqrt{\lambda_{int}}$ respectively. The directions of the maximum, minimum and intermediate velocities, or the *principal* velocities, are related to the eigenvectors of $[g]$.

- The principal velocities are defined as

$$\mathbf{v}_i = [J(\boldsymbol{\psi})]\dot{\boldsymbol{\theta}}_i, \quad i = 1, 2, 3 \tag{22}$$

  where $\dot{\boldsymbol{\theta}}_i$ is the eigenvector corresponding to eigenvalue λ_i. They can also be expressed in the form

$$\mathbf{v}_i = \sqrt{\lambda_i}\mathbf{x}_i \quad i = 1, 2, 3 \tag{23}$$

  where $\mathbf{x}_i$ in an *unit vector* along $[J(\boldsymbol{\psi})]\dot{\boldsymbol{\theta}}_i$.

These three vectors form the three principal axes of the ellipsoid and determine the shape of the ellipsoid(for an ellipse there are only a maximum and a minimum). If the normalization $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = k^2$ is used, then the maximum and minimum values are scaled by k but the shape of the velocity ellipsoid(or ellipse) doesn't change. The elemental volume(area in case of a surface) is proportional to $k\sqrt{\det[g]}$

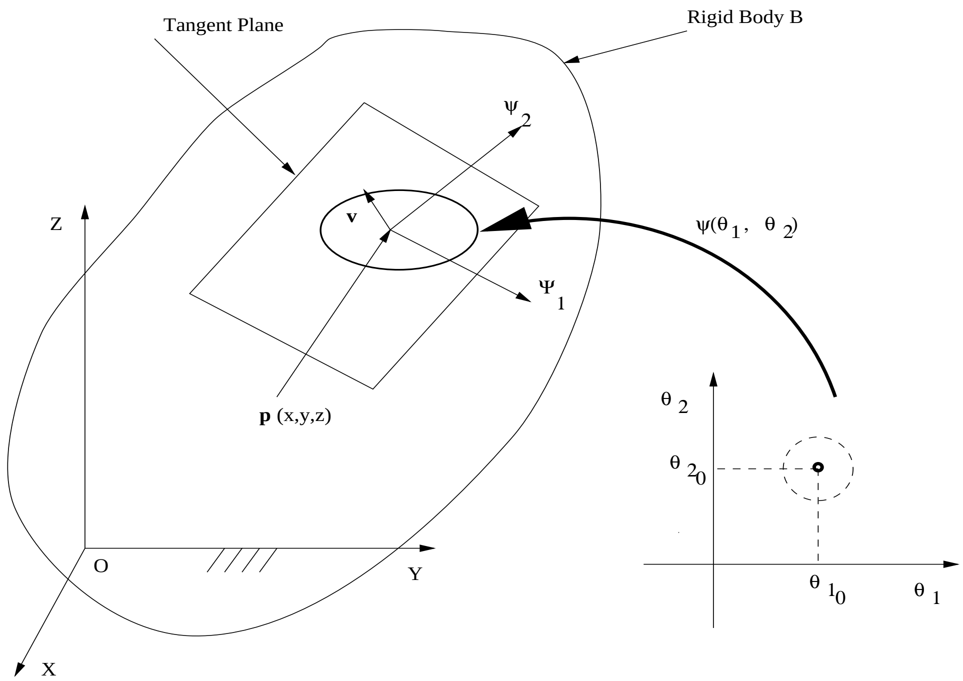

Figure 1: The velocity ellipse in the tangent plane

- The velocity vector, $\mathbf{v}$, or the elements of the matrix $[g]$ determine the first order properties of the point trajectories. For the second-order properties we consider the acceleration vector given in terms of the first and second partial derivatives of $\boldsymbol{\psi}$ as

$$\mathbf{a} = \sum_{i=1}^{n} \boldsymbol{\psi}_i \ddot{\theta}_i + \sum_{i,j=1}^{n} \boldsymbol{\psi}_{ij} \dot{\theta}_i \dot{\theta}_j \tag{24}$$

When the number of independent θ_i is two and the point trajectory is in $\Re^3$, we get from differential geometry,

$$\mathbf{a} = \sum_{i=1}^{2} \boldsymbol{\psi}_i \ddot{\theta}_i + \sum_{i,j=1}^{n} (\Gamma_{ijk} \boldsymbol{\psi}_k + L_{ij}\mathbf{n}) \dot{\theta}_i \dot{\theta}_j \tag{25}$$

where $\mathbf{n}$ is the normal vector, L_{ij}'s are the dot products $\boldsymbol{\psi}_{ij} \cdot \mathbf{n}, i, j = 1, 2$, and the six

Γ_{ijk}'s[4] are called the *Christoffel symbols*[26] of the first kind, and are given as

$$\Gamma_{ijk} = \boldsymbol{\psi}_{ij} \cdot \boldsymbol{\psi}_k \tag{26}$$

The six *Christoffel symbols* determine the nature of the curves *on* the surface and a curve is said to be a *geodesic*[5] if the geodesic curvature is zero. It can be shown [26] that the geodesic curvature is zero if the tangential acceleration is zero, and if $\theta_1(t)$, $\theta_2(t)$ satisfies the non-linear differential equations

$$\ddot{\theta}_k + \sum_{i,j=1}^{2} \Gamma_{ij}^k \dot{\theta}_i \dot{\theta}_j = 0, \quad k = 1, 2 \tag{27}$$

the point trajectory is a geodesic. In the above equation, Γ_{ij}^k's are called the Christoffel symbols of the second kind, and are given by $\sum_{l=1}^{2}(\boldsymbol{\psi}_{ij} \cdot \boldsymbol{\psi}_l) g^{lk}, \;\; i, j, k = 1, 2$, and g^{lk} is the (l, k) element of $[g]^{-1}$.

The local geometry of the surface is determined by the *Gaussian curvature* given by

$$K = \frac{\det[L]}{\det[g]} \tag{28}$$

The Gaussian curvature and the Christoffel symbols can also be derived purely in terms of the metric coefficients, g_{ij} and their first partial derivatives[26]. The second order properties of a surface are completely determined by the elements g_{ij}, L_{ij}, and Γ_{ij}^k.

- If the number of independent θ_i is three and the motion is in 3D space, then there is no notion of a normal vector. In such a case, we get 18 Christoffel symbols, which together with the 6 g_{ij}'s completely determine the second-order properties of a surface.

In the next subsection, we extend the above mentioned, well-known differential-geometric notions of a point trajectory to that of the entire rigid body motion by the use of dual numbers and vectors.

3.2 Rigid body displacement

A general rigid body displacement can be expressed as a 4×4 matrix of homogeneous coordinates[10] or as 3×3 dual orthogonal matrices[20]. By using the properties of these

[4] By the property of partial differentiation, $\Gamma_{ijk} = \Gamma_{jik}$ and there are only 6 Christoffel symbols.

[5] A geodesic is the *shortest* distance between two points on a surface.

matrices, and differentiating the matrix elements with respect to time, one can obtain expressions for left- and right-invariant velocities of a moving rigid body. For our purpose of studying instantaneous rigid body kinematics, we use the well known form of expressing the angular velocity and linear velocity of a point on the rigid body as a dual vector[17],

$$\hat{\mathcal{V}} = \boldsymbol{\omega} + \epsilon \mathbf{v}_p = \sum_{i=1}^{n} \hat{\mathcal{S}}_i \dot{\theta}_i \tag{29}$$

where $\boldsymbol{\omega}$, $\mathbf{v}_p$ are the angular velocity and linear velocity of a point respectively, $\hat{\mathcal{S}}_i,\ i = 1, 2, .., n$ are n independent screws expressed using dual vectors, and $\dot{\theta}_i,\ i = 1, 2, ..., n$ are the time derivatives of the n motion parameters. The above equation can also be written in terms of the well known dual Jacobian matrix as

$$\hat{\mathcal{V}} = [\hat{J}]\dot{\boldsymbol{\theta}} \tag{30}$$

where $\dot{\boldsymbol{\theta}}$ is the vector $(\dot{\theta}_1, ..., \dot{\theta}_n)^T$ and the i^{th} column of $[\hat{J}]$ is the screw $\hat{\mathcal{S}}_i$.

Using the inner product between two screws (see equation (6)), we can write

$$< \hat{\mathcal{V}}, \hat{\mathcal{V}} >= \dot{\boldsymbol{\theta}}^T [\hat{g}] \dot{\boldsymbol{\theta}} \tag{31}$$

where the matrix elements $\hat{g}_{ij}$ are the inner products $< \hat{\mathcal{S}}_i, \hat{\mathcal{S}}_j >,\ i, j = 1, 2, .., n$. The elements of the matrix $[\hat{g}]$ are dual numbers and the matrix $[\hat{g}]$ is symmetric and positive semi-definite. We make the following observations, analogous to point trajectories, from the definition of $[\hat{g}]$ and $[\hat{J}]$:

- If $[\hat{g}]$ is non-singular(i.e., $\det[\hat{g}] \neq 0$), then we can write

  $$\hat{\mathcal{V}}^T ([\hat{J}][\hat{g}]^{-1})([\hat{J}][\hat{g}]^{-1})^T \hat{\mathcal{V}} = \dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} \tag{32}$$

  The matrix $([\hat{J}][\hat{g}]^{-1})([\hat{J}][\hat{g}]^{-1})^T$ is symmetric and positive semi-definite, and for a constraint of the form $\dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} = 1$, the tip of the dual velocity vector $\hat{\mathcal{V}}$, in analogy with point trajectories, lies on a dual ellipsoid. For two-degree-of-freedom motions the tip of the dual vector $\hat{\mathcal{V}}$, in analogy with point trajectories, lies on a dual ellipse[6].

- The principal values of $\hat{\mathcal{V}}^2$ subject to constraint $\dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} = 1$ can be obtained by solving the eigenvalue problem

  $$[\hat{g}]\dot{\boldsymbol{\theta}} - \hat{\lambda}\dot{\boldsymbol{\theta}} = 0 \tag{33}$$

[6] The dual ellipse has been identified as a cylindroid in [28].

The principal velocities are obtained as

$$\hat{\mathcal{V}}_i = [\hat{J}]\dot{\boldsymbol{\theta}}_i, \quad i = 1, 2, 3 \tag{34}$$

where $\dot{\boldsymbol{\theta}}_i$ is the eigenvector corresponding to the eigenvalue $\hat{\lambda}_i$. The principal velocities can also be written, in analogy with equation (23), as

$$\hat{\mathcal{V}}_i = \sqrt{\hat{\lambda}_i}\hat{\mathbf{x}}_i, \quad i = 1, 2, 3 \tag{35}$$

where $\hat{\mathbf{x}}_i$ is the i^{th} dual eigenvector of $[\hat{J}][\hat{J}^T]$ and is given by $\mathbf{x}_i + \epsilon\mathbf{x}_{0i}$.

Expanding $\hat{\lambda}_i$ and $\hat{\mathbf{x}}_i$ in terms of real and dual parts and simplifying, we get

$$\hat{\mathcal{V}}_i = \sqrt{\lambda_i}(\mathbf{x}_i + \epsilon(\frac{\lambda_{0i}}{2\lambda_i}\mathbf{x}_i + \mathbf{x}_{0i})), \quad \lambda_i \neq 0 \tag{36}$$

Comparing with equation (8), we find

$$\begin{aligned} \boldsymbol{\omega}_i &= \sqrt{\lambda_i}\mathbf{x}_i & (37) \\ \mathbf{v}_i &= \sqrt{\lambda_i}(h_i^*\mathbf{x}_i + \mathbf{x}_{0i}) & (38) \end{aligned}$$

where the i^{th} principal pitch is given by

$$h_i^* = \frac{\lambda_{0i}}{2\lambda_i} \tag{39}$$

For a two-degree-of-freedom motion the elements of the matrix $[\hat{g}]$, in terms of the pitches(h_1, h_2), the angle, ϕ, and the distance, d, between the two screws, $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$, are

$$\begin{aligned} \hat{g}_{11} &= 1 + \epsilon(2h_1) \\ \hat{g}_{12} = \hat{g}_{21} &= \cos\phi + \epsilon((h_1 + h_2)\cos\phi - d\sin\phi) \\ \hat{g}_{22} &= 1 + \epsilon(2h_2) \end{aligned} \tag{40}$$

The eigenvalues are given by

$$\begin{aligned} \hat{\lambda}_1 &= 2\cos^2\phi/2(1 + \epsilon((h_1 + h_2) - d\tan(\phi/2)) \\ \hat{\lambda}_2 &= 2\sin^2\phi/2(1 + \epsilon((h_1 + h_2) + d\cot(\phi/2)) \end{aligned} \tag{41}$$

The magnitudes of principal angular velocities are given as

$$\begin{aligned} |\boldsymbol{\omega}_1| &= \sqrt{\lambda_1} = \sqrt{2}\cos(\phi/2) \\ |\boldsymbol{\omega}_2| &= \sqrt{\lambda_2} = \sqrt{2}\sin(\phi/2) \end{aligned} \tag{42}$$

The principal pitches are given by

$$
\begin{aligned}
h_1^* &= 1/2((h_1+h_2) - d\tan(\phi/2)) \\
h_2^* &= 1/2((h_1+h_2) + d\cot(\phi/2))
\end{aligned} \tag{43}
$$

The determinant, $\det[\hat{g}]$, is given by

$$
\det[\hat{g}] = \sin^2\phi(1+\epsilon 2(h_1+h_2+d\cot\phi)) \tag{44}
$$

It may be noted that $\det[\hat{g}]$ is zero if $\phi = 0, n\pi$, i.e., the axis of the two screws are parallel. In such a case the two screws are not independent and we have a one-degree-of-freedom motion of the rigid body – $|\boldsymbol{\omega}_2|$ is zero from equation (42).

For a three-degree-of-freedom motion of the rigid body, the matrix $[\hat{g}]$ is 3×3 and it has three eigenvalues. The maximum, minimum, and intermediate values of $|\hat{\mathcal{V}}|$ are the square roots of the three eigenvalues and are along the three principal axes of the dual ellipsoid.

As in the case of point trajectories, it may be noted that the normalization $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = k^2$ scales the eigenvalues without changing the shape of the dual velocity ellipsoid(or ellipse). The elemental dual area is proportional to $\sqrt{\det[\hat{g}]}$. For a two-degree-of-freedom motion we have

$$
\sqrt{\det[\hat{g}]} = \sin\phi(1+\epsilon(h_1+h_2+d\cot\phi)) \tag{45}
$$

- To study the *second-order* properties of the rigid body motion, in analogy with the point trajectories, we define a dual acceleration vector. We get by differentiating equation (29),

$$
\hat{a} = \sum_{i=1}^{3}\hat{\mathcal{S}}_i\ddot{\theta}_i + \sum_{i,j=1}^{3}\hat{\mathcal{S}}_{ij}\dot{\theta}_i\dot{\theta}_j \tag{46}
$$

where $\hat{\mathcal{S}}_{ij} = \frac{\partial\mathcal{S}_i}{\partial\theta_j} + \epsilon(\frac{\partial\mathcal{S}_{0i}}{\partial\theta_j} + \mathcal{S}_{0i}\times\mathcal{S}_j)$.

The dual components $\hat{a}_l,\ l=1,2,3$, are obtained by taking the dot product of $\hat{\mathcal{S}}_l$ and $\hat{a}$, and we get

$$
\hat{a}_l = \sum_{i=1}^{3}\hat{g}_{il}\ddot{\theta}_j + \sum_{i,j=1}^{2}\hat{\Gamma}_{ijl}\dot{\theta}_i\dot{\theta}_j \qquad l=1,2,3 \tag{47}
$$

where $\hat{g}_{il} = \hat{\mathcal{S}}_i \cdot \hat{\mathcal{S}}_l$ and $\hat{\Gamma}_{ijl} = \hat{\mathcal{S}}_{ij} \cdot \hat{\mathcal{S}}_l$. The symbols $\hat{\Gamma}_{ijl}$, in analogy to point trajectories, are called the dual Christoffel symbols of the first kind.

The dual geodesic can be obtained by setting the dual tangential acceleration to zero. We get two dual non-linear differential equations of the form

$$0 = \sum_{i=1}^{2} \hat{g}_{il}\ddot{\theta}_i + \sum_{i,j=1}^{2} \hat{\Gamma}_{ijl}\dot{\theta}_i\dot{\theta}_j, \quad l = 1, 2 \tag{48}$$

A geometric interpretation of zero dual tangential acceleration can be obtained by differentiating the dual velocity vector. Differentiating $\hat{\mathcal{V}}$ in equation (7), we have [17]

$$\hat{\mathbf{a}} = \boldsymbol{\alpha} + \epsilon(\mathbf{a}_p + \mathbf{v}_p \times \boldsymbol{\omega}) \tag{49}$$

where $\boldsymbol{\alpha}$ is the angular acceleration of the rigid body and $\mathbf{a}_p$ is the linear acceleration of a point on the rigid body. For three-degree-of-freedom motion, zero dual tangential acceleration leads to

$$\begin{aligned} \boldsymbol{\alpha} &= \mathbf{0} \\ \mathbf{a}_p + \mathbf{v}_p \times \boldsymbol{\omega} &= \mathbf{0} \end{aligned} \tag{50}$$

If $\boldsymbol{\alpha}$ is zero, then the angular velocity $\boldsymbol{\omega}$ is constant, and by taking dot product of second equation with $\boldsymbol{\omega}$ we get $\mathbf{a}_p \cdot \boldsymbol{\omega} = 0$. Differentiating the expression of pitch, h, given in equation (9), with respect to time, t, and noting that $\boldsymbol{\omega}$ is constant, we get

$$\frac{d\,h}{d\,t} = \frac{1}{|\boldsymbol{\omega}|^2}\,\boldsymbol{\omega} \cdot \mathbf{a}_p = 0 \tag{51}$$

Hence a geodesic motion with zero dual tangential acceleration corresponds to a constant angular velocity and constant pitch motion.

In the next section, we apply the results to the analysis of serial and parallel manipulators.

4 Illustrative Examples

In this section, we illustrate the concepts developed above by means of two examples, namely that of a serial 2R manipulator and a parallel three-degree-of-freedom RPSSPR-SPR manipulator. In the first example we show that the well known cylindroid (see, for example, Roth 1984) is the *same* at every point of the workspace. For the second order property, we show that all the dual Christoffel symbols have real part as zero. To obtain, better insight, we plot the surface traced by the central cylindroid axis and the geodesic. In the second example, we show that the developed concepts can be applied to the analysis of parallel manipulators.

4.1 A 2R manipulator

Figure 2 shows a two-degree-of-freedom manipulator with revolute(R) joints. The joint variables are θ_1 and θ_2. In terms of the link lengths, a_{ij}'s, link offsets, d_i's, and twists α_{ij}'s, the mapping function $\boldsymbol{\psi}$, for a point $(x,y,z)^T$ on the moving rigid body, can be written as

$$(x,y,z)^T = \boldsymbol{\psi}(\theta_1,\theta_2) = d_1 S_1 + a_{12} a_{12} + d_2 \mathbf{S}_2 + a_{23}\mathbf{a}_{23} \tag{52}$$

where[7]

$$\begin{aligned} S_1 &= (0,0,1)^T \\ a_{12} &= (c_1, s_1, 0)^T \\ S_2 &= (s_1 s\alpha_{12}, -c_1 s\alpha_{12}, c\alpha_{12})^T \\ a_{23} &= [(c_1c_2 - s_1s_2c\alpha_{12}), (s_1c_2 + c_1s_2c\alpha_{12}), s\alpha_{12}s_2]^T \end{aligned} \tag{53}$$

The partial derivatives $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are given by

$$\begin{aligned} \boldsymbol{\psi}_1 &= a_{12}(-s_1, c_1, 0)^T + d_2(s\alpha_{12}c_1, s\alpha_{12}s_1, 0)^T + a_{23}(-s_1c_2 - c_1s_2c\alpha_{12}, c_1c_2 - s_1s_2c\alpha_{12}, 0)^T \\ \boldsymbol{\psi}_2 &= a_{23}(-c_1s_2 - s_1c_2c\alpha_{12}, -s_1s_2 + c_1c_2c\alpha_{12}, s\alpha_{12}c_2)^T \end{aligned} \tag{54}$$

The linear velocity of the point $p(x,y,z)$ is given by $\boldsymbol{\psi}_1\dot{\theta}_1 + \boldsymbol{\psi}_2\dot{\theta}_2$, and the angular velocity of the rigid body is given by $S_1\dot{\theta}_1 + \mathbf{S}_2\dot{\theta}_2$. The dual velocity vector is defined as

$$\hat{\mathcal{V}} = \omega + \epsilon v = \sum_{i=1}^{2} \hat{\mathcal{S}}_i \dot{\theta}_i \tag{55}$$

where $\hat{\mathcal{S}}_i = S_i + \epsilon\boldsymbol{\psi}_i$ with $S_i,\ i=1,2$ and $\boldsymbol{\psi}_i, i=1,2$ defined in equation (53) and (54) respectively. The elements of the symmetric and positive definite dual matrix $\hat{g}$ are

$$\begin{aligned} \hat{g}_{11} &= 1 \\ \hat{g}_{12} &= \hat{g}_{21} = c\alpha_{12} + \epsilon(-a_{12}s\alpha_{12}) \\ \hat{g}_{22} &= 1 \end{aligned} \tag{56}$$

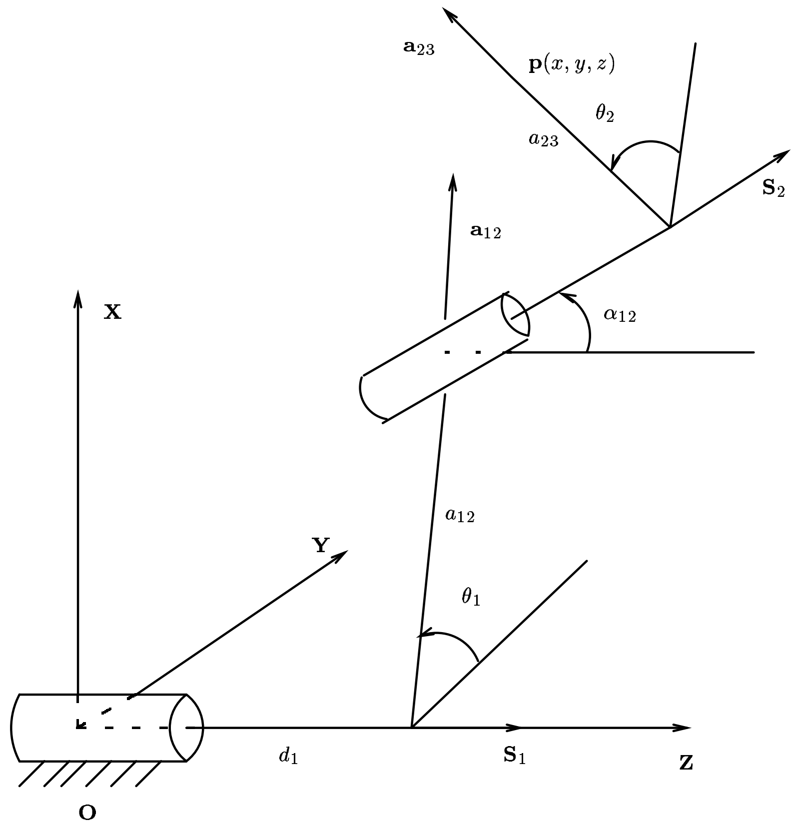

Figure 2: A spatial 2R manipulator

and the determinant of $[\hat{g}]$ is $\sin^2\alpha_{12} + \epsilon a_{12}\sin 2\alpha_{12}$.

The eigenvalues of $[\hat{g}]$ are given by

$$\begin{aligned}\hat{\lambda}_1 &= 2\sin^2(\alpha_{12}/2) + \epsilon a_{12}\sin\alpha_{12} \\ \hat{\lambda}_2 &= 2\cos^2(\alpha_{12}/2) - \epsilon a_{12}\sin\alpha_{12}\end{aligned} \tag{57}$$

The principal pitches are computed from equation (39) as

$$\begin{aligned}h_1^* &= 0.5\ a_{12}\tan(\alpha_{12}/2) \\ h_2^* &= -0.5\ a_{12}\cot(\alpha_{12}/2)\end{aligned} \tag{58}$$

We make the following observations on equations (56,57,58):

- The coefficients of the matrix $[\hat{g}]$ are independent of the motion parameters, θ_i, and they depend only on the architecture of the mechanism.
- The eigenvalues of $[\hat{g}]$ and, hence, the principal pitches are constant quantities.
- If the angle of twist between the screws, $\alpha_{12} = 0, \pi$, we have $\det[\hat{g}] = 0$ and the motion is singular. The screw axes are parallel in this case.

The above observations imply that the dual ellipse, and the cylindroid for the 2R manipulator will be geometrically the same *at all configurations*, as they depend only on the architectural parameters. Hence the first-order properties will be the same at all points in the workspace. A representative cylindroid is shown in figure 3 for the principal pitches $h_1^* = 3.0$ and $h_2^* = 5.0$.

Second Order properties

We first observe that all the elements of $[\hat{g}]$ are independent of the motion parameters θ_1 and θ_2. This implies that the partial derivatives, with respect to the motion parameters, of all the g_{ij}'s are zero. To get better insight, we describe the second-order properties in terms of the motion of the cylindroid axis. This is analogous to the study of second-order properties of a surface by considering the variation in the normal vector at different points on the surface.

[7] We use the symbols $s_{(\cdot)}$ and $c_{(\cdot)}$ to denote $\sin(\cdot)$ and $\cos(\cdot)$ respectively throughout this paper.

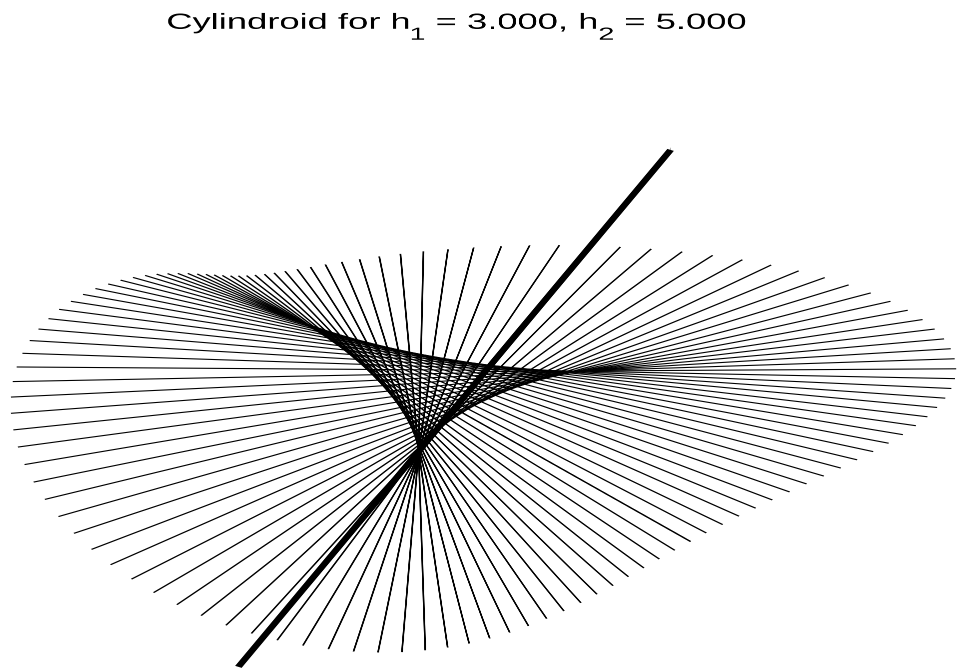

Figure 3: The cylindroid of a 2R manipulator

The principal screws are obtained from equation (34) as

$$\begin{aligned}\hat{\mathcal{V}}_1 &= \frac{1}{\sqrt{2}}(\hat{\mathcal{S}}_1 - \hat{\mathcal{S}}_2)\\ \hat{\mathcal{V}}_2 &= \frac{1}{\sqrt{2}}(\hat{\mathcal{S}}_1 + \hat{\mathcal{S}}_2)\end{aligned} \tag{59}$$

The cylindroid axis is given by $\hat{\mathcal{S}} = \hat{\mathcal{V}}_1 \times \hat{\mathcal{V}}_2$, where the real and dual parts of $\hat{\mathbf{S}}$ are given by

$$\begin{aligned}\mathbf{S} &= (\cos\theta_1 \sin\alpha_{12}, \sin\alpha_{12}\sin\theta_1, 0)^T\\ \mathbf{S}_0 &= (\cos\alpha_{12}(a_{12}\cos\theta_1 + d_2\sin\alpha_{12}\sin\theta_1) + a_{23}\sin^2\alpha_{12}\sin\theta_1\sin\theta_2,\\ &\quad \cos\alpha_{12}(a_{12}\sin\theta_1 - d_2\cos\theta_1\sin\alpha_{12}) - a_{23}\cos\theta_1\sin^2\alpha_{12}\sin\theta_2,\\ &\quad \sin\alpha_{12}(-d_2\sin\alpha_{12} + a_{23}\cos\alpha_{12}\sin\theta_2))^T\end{aligned} \tag{60}$$

The foot of the perpendicular to the cylindroid axis is obtained from equation(3) as

$$\begin{aligned}(x,y,z)^T = \mathbf{r}_0 &= (\ \sin\theta_1(-d_2\sin\alpha_{12} + a_{23}\cos\alpha_{12}\sin\theta_2),\\ &\quad \cos\theta_1(d_2\sin\alpha_{12} + a_{23}\cos\alpha_{12}\sin\theta_2),\\ &\quad - d_2\cos\alpha_{12} + a_{23}\sin\alpha_{12}\sin\theta_2\ )^T\end{aligned} \tag{61}$$

Eliminating θ_1 and θ_2 from the above vector equation, we find that the locus of the foot of the perpendicular is given by

$$(x^2 + y^2)\sin^2\alpha_{12} - (d_2 + z\cos\alpha_{12})^2 = 0 \tag{62}$$

From equations (61) and (62), we can infer the following regarding the motion of the cylindroid axis:

- The locus of the cylindroid axis for 2R manipulator is a two parameter family of lines parallel to the **XY** plane. The envelope of this family is given by a *quadratic cone* described by the foot of the perpendicular from the origin $\mathbf{r}_0$. The axis of the cone is along the **Z** axis.

- The effect of variation in the parameter θ_1 is a rotation of the cylindroid axis about the **Z** axis, whereas changing θ_2 results in the translation of the axis in the **Z** direction.

The locus of the two parameter family of lines, representing the motion of cylindroid axis for the 2R manipulator, is shown in figure 4. For clarity, we have shown the family of lines for *only* three values of θ_1 with line segments. The locus of the foot of the perpendicular is shown in figure 5. The numerical values used in these plots are $a_{12} = 1.0, a_{23} = 1.0, d_2 = 0.5, \alpha_{12} = \pi/4$.

Geodesic Motion

As mentioned above, the $\hat{g}_{ij}$'s are independent of θ_1 and θ_2 and all $\hat{\Gamma}_{ijk}$'s have real part as zero. The $\hat{\Gamma}_{ijk}$'s are given by

$$\begin{aligned}
\hat{\Gamma}_{111} &= 0 \\
\hat{\Gamma}_{121} &= \epsilon(s\alpha_{12}(-d_2 s\alpha_{12} + a_2 c_2 c\alpha_{12})) \\
\hat{\Gamma}_{211} &= \epsilon(s\alpha_{12}(d_2 s\alpha_{12} - a_2 c_2 c\alpha_{12})) \\
\hat{\Gamma}_{221} &= 0 \\
\hat{\Gamma}_{112} &= 0 \\
\hat{\Gamma}_{122} &= \epsilon(a_2 s\alpha_{12} s_2) \\
\hat{\Gamma}_{212} &= \epsilon(-a_2 s\alpha_{12} s_2) \\
\hat{\Gamma}_{222} &= 0
\end{aligned} \tag{63}$$

It can be observed that $\hat{\Gamma}_{211} = -\hat{\Gamma}_{121}$ and $\hat{\Gamma}_{212} = -\hat{\Gamma}_{122}$. The dual geodesic equation, given in(48), therefore reduces to the second-order linear differential equation of the form

$$\ddot{\boldsymbol{\theta}} = \mathbf{0} \tag{64}$$

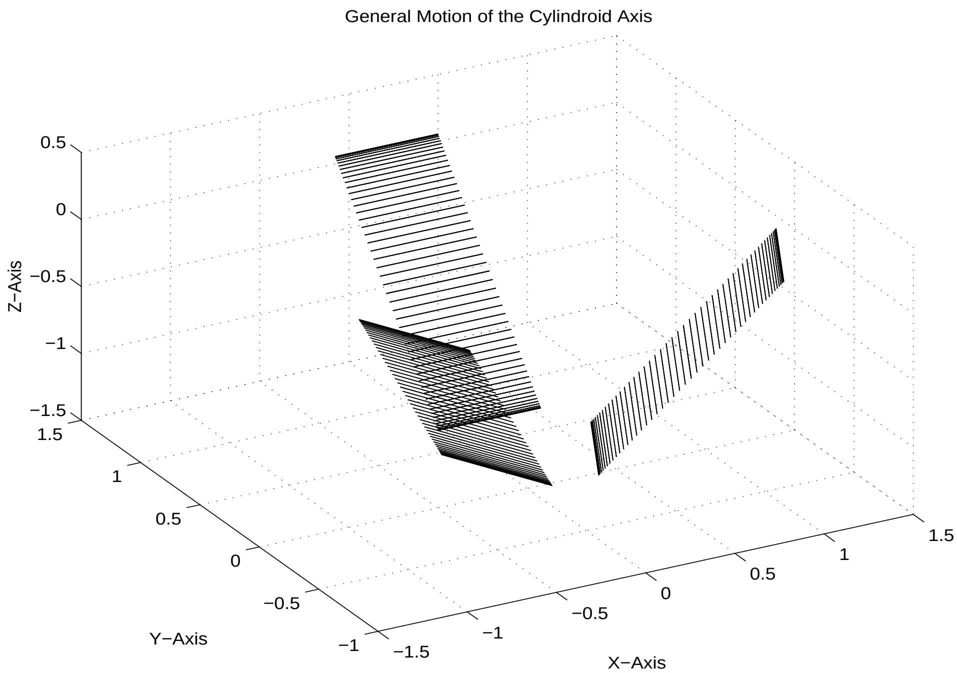

Figure 4: Locus of the central axis of the cylindroid

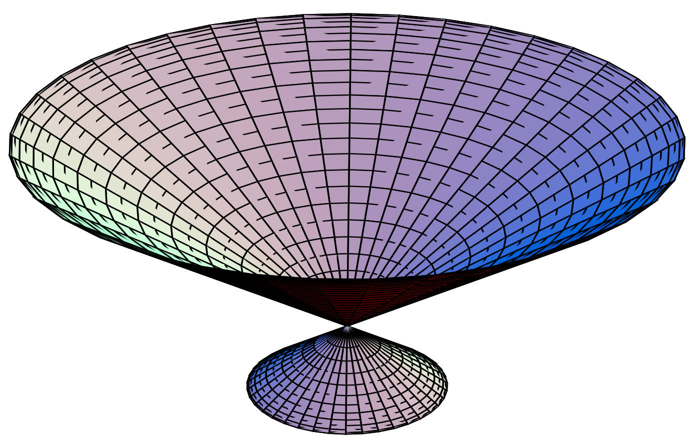

Figure 5: Locus of the foot of the perpendicular from the origin to the cylindroid axis

The solution of the above equation is of the form $\theta_i(t) = \dot{\theta}_i(0)t + \theta_i(0),\ i = 1, 2$, where $\dot{\theta}_i(0)$ and $\theta_i(0)$ denote the initial values of the velocity and position in the $i^{\rm th}$ parameter, and t denotes time. Eliminating the independent parameter t from the expressions of θ_i, we can write

$$\theta_2 = c_1\theta_1 + c_2 \tag{65}$$

where $c_1 = -\frac{\dot{\theta}_2(0)}{\dot{\theta}_1(0)}, c_2 = \frac{\dot{\theta}_2(0)\theta_1(0) - \dot{\theta}_1(0)\theta_2(0)}{\dot{\theta}_1(0)}$.

The last equation shows that the geodesics are straight lines in the configuration space. The geodesic motion of the cylindroid axis with $c_1 = 2$ and $c_2 = 0$ is shown in figure 6. It may be mentioned that the ruled surface shown in figure 6 is a one parameter subset of the locus of the cylindroid axis shown in figure 4. Figure 7 shows the locus of the foot of the perpendicular for $c_1 = 2$ and $c_2 = 0$. It may be mentioned that this curve lies on the quadratic cone shown in figure 5. It is to be noted that the shape of the geodesic curve or the locus of $\mathbf{r}_0$ for geodesic motion, is determined by quantity c_1, which is a function of the initial conditions. The curve is closed for rational values of c_1 and open for irrational values(see figure 8). Moreover, it contains $2c_1$ loops, if c_1 is an integer(see figure 7). The effect of c_2 is to rotate the curve about the $\mathbf{Z}$ axis, which is the axis of symmetry of the curve. Figure 9 shows the $\mathbf{XY}$ projection of five geodesic curves for $c_1 = 2$ and $c_2 = 1, 2, .., 5$.

4.2 A three-degree-of-freedom parallel manipulator

In this section, we show that the theoretical analysis developed in the previous sections is also applicable to the analysis of parallel manipulators. We use the three-loop, three-degree-of-freedom RPSSPR-SPR mechanism of figure 10 described in [29] for this purpose.

The geometry chosen is same as in reference [29] where the revolute joints axes are assumed to be co-planar and are perpendicular to the medians passing through the respective vertices. Assuming that the length of the medians in the base equilateral triangle are unity, we can obtain the coordinates of the of three spherical joints in the fixed coordinate system. These are given by

$$\begin{aligned}
\mathbf{P}_1 &= [(1 - l_1c_1), 0, l_1s_1]^T \\
\mathbf{P}_2 &= [-0.5(1 - l_2c_2), \sqrt{3}/2(1 - l_2c_2), l_2s_2]^T \\
\mathbf{P}_3 &= [-0.5(1 - l_3c_3), -\sqrt{3}/2(1 - l_3c_3), l_3s_3]^T
\end{aligned} \tag{66}$$

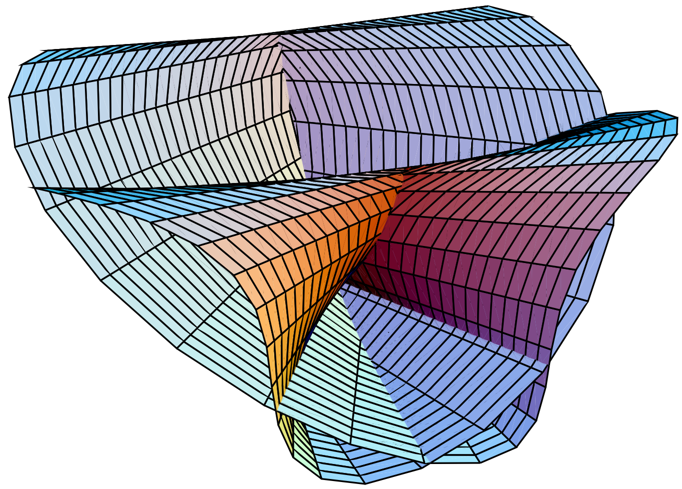

Figure 6: Locus of the cylindroid axis for geodesic motion

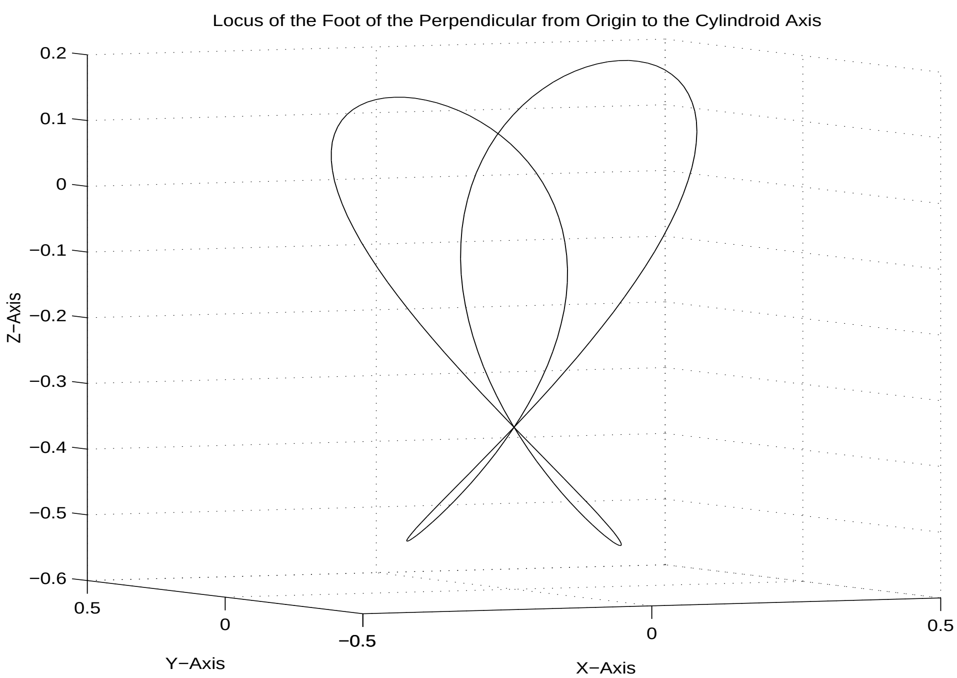

Figure 7: Locus of $\mathbf{r}_0$ for geodesic motion

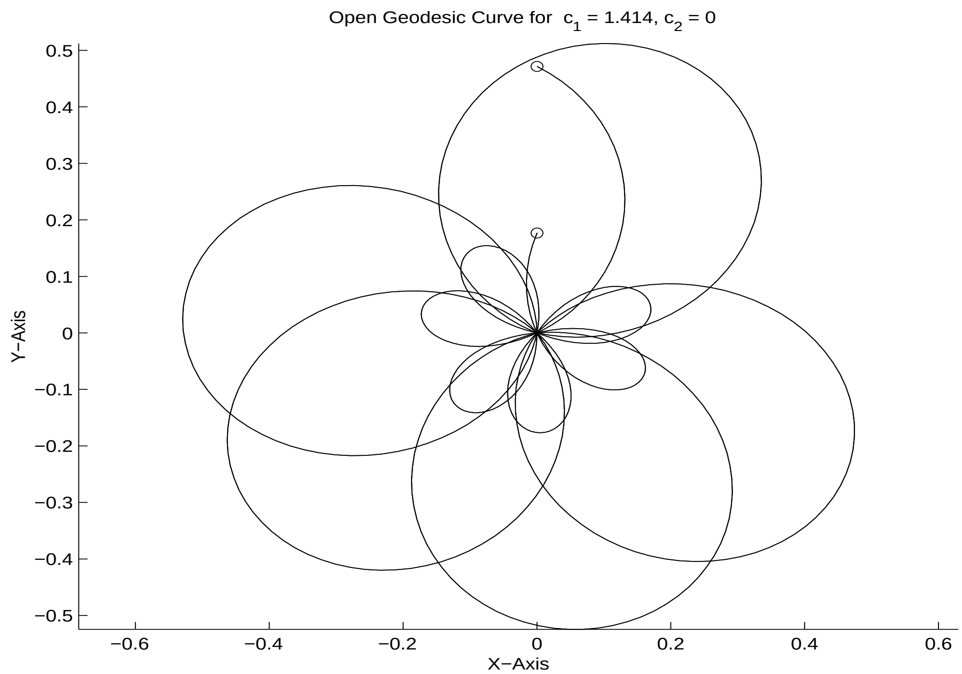

Figure 8: Locus of $\mathbf{r}_0$ for geodesic motion: Open curve for irrational c_1

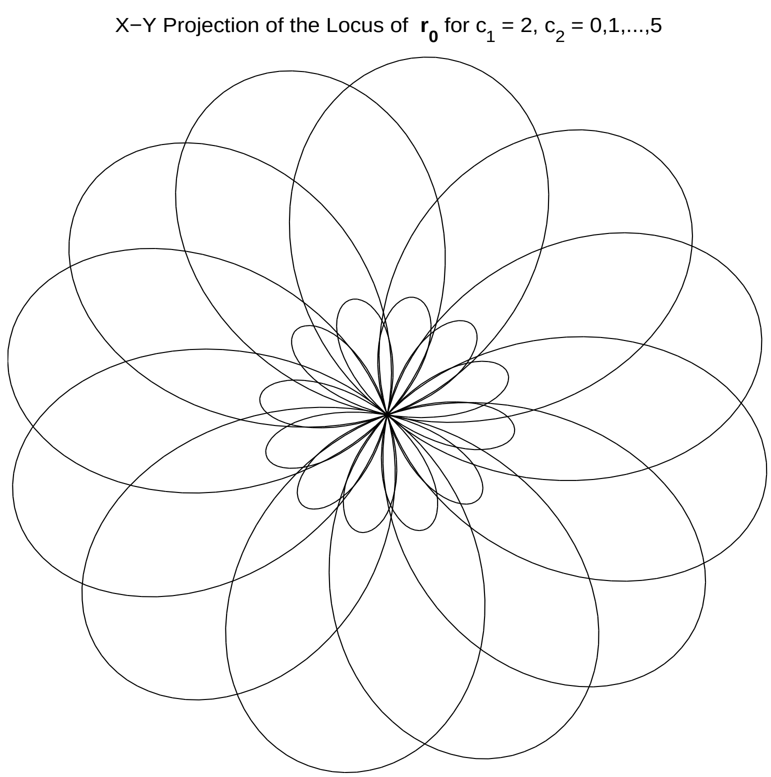

Figure 9: Locus of $\mathbf{r}_0$ for geodesic motion: Effect of variation of c_2

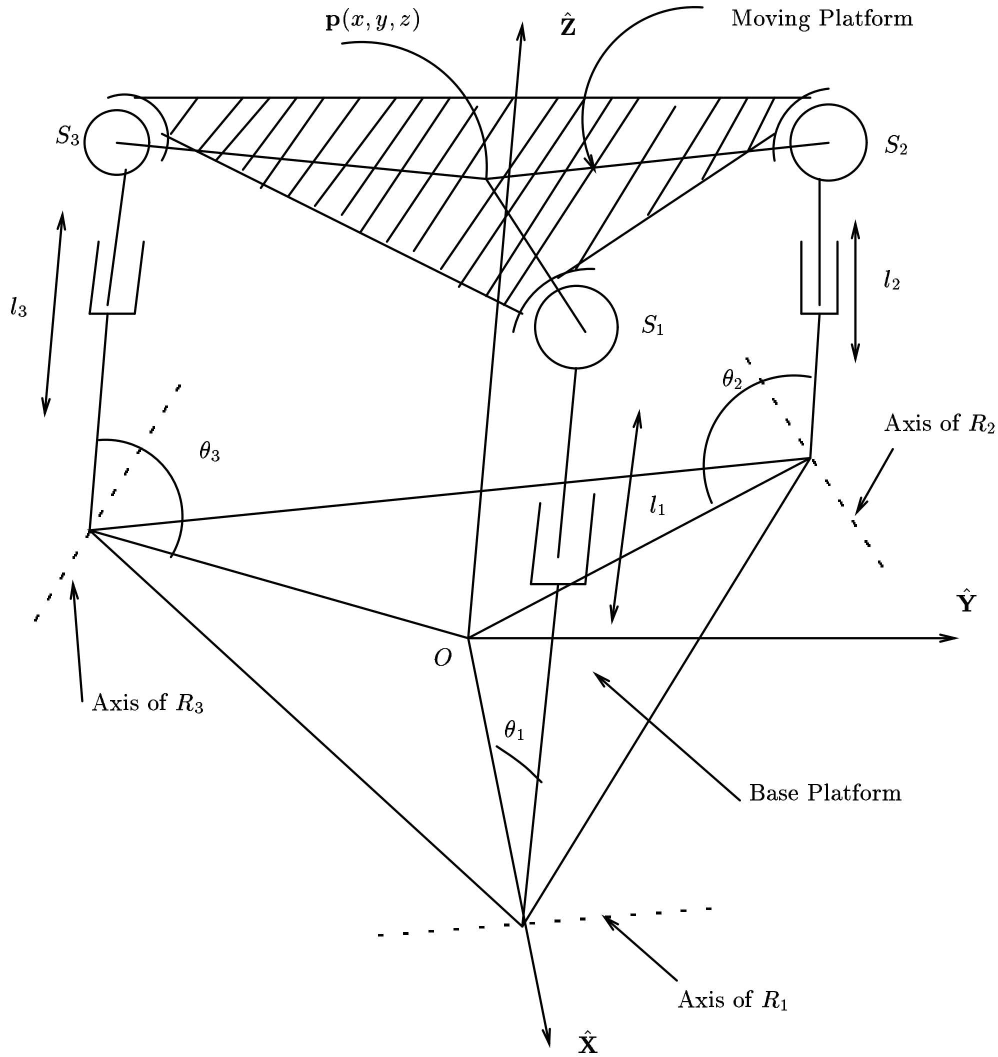

Figure 10: The RPSSPR-SPR parallel manipulator

where $\theta_i, i = 1, 2, 3$ are rotations at the three passive rotary joints and $l_i, i = 1, 2, 3$ are the translations at the actuated prismatic joints.

The loop closure equations are obtained from the fact that the distance between the spherical joints are constant and are of the form

$$(\mathbf{P}_i - \mathbf{P}_j) \cdot (\mathbf{P}_i - \mathbf{P}_j) = k_{ij}^2, \qquad i, j = 1, 2, 3, i \neq j \tag{67}$$

where k_{ij} is the distance between the spherical joints i and j respectively.

Differentiating the three constraint equations with respect to time, we get

$$\begin{pmatrix} 3l_1s_1 - l_1l_2s_1c_2 - 2l_1l_2c_1s_2 \\ 0 \\ 3l_1s_1 - l_1l_3s_1c_3 - 2l_1l_3c_1s_3 \end{pmatrix} \dot{\theta}_1 \; + \; \begin{pmatrix} 3l_2s_2 - l_1l_2c_1s_2 - 2l_1l_2s_1c_2 \\ 3l_2s_2 - l_2l_3s_2c_3 - 2l_2l_3c_2s_3 \\ 0 \end{pmatrix} \dot{\theta}_2 \, +$$
$$\begin{pmatrix} 0 \\ 3l_3s_3 - l_2l_3c_2s_3 - 2l_2l_3s_2c_3 \\ 3l_3s_3 - l_1l_3c_1s_3 - 2l_1l_3s_1c_3 \end{pmatrix} \dot{\theta}_3 \; + \; \begin{pmatrix} 2l_1 - 3c_1 + l_2c_1c_2 - 2l_2s_1s_2 \\ 0 \\ 2l_1 - 3c_1 + l_3c_1c_3 - 2l_3s_1s_3 \end{pmatrix} \dot{l}_1 \, +$$
$$\begin{pmatrix} 2l_2 - 3c_2 + l_1c_1c_2 - 2l_1s_1s_2 \\ 2l_2 - 3c_2 + l_3c_2c_3 - 2l_3s_2s_3 \\ 0 \end{pmatrix} \dot{l}_2 \; + \; \begin{pmatrix} 0 \\ 2l_3 - 3c_3 + l_2c_2c_3 - 2l_2s_2s_3 \\ 2l_3 - 3c_3 + l_1c_1c_3 - 2l_1s_1s_3 \end{pmatrix} \dot{l}_3 = \mathbf{0}$$

The above equation can be written in the form

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = -[K^*]^{-1}[K] \begin{pmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{l}_3 \end{pmatrix} \tag{68}$$

where the columns of $[K^*]$ and $[K]$ are coefficients of $\dot{\theta}_i, i = 1, 2, 3$ and $\dot{l}_i, i = 1, 2, 3$ respectively.

Assuming all the lengths k_{ij}'s are $\sqrt{3}/2$ (the lengths of the medians of the top platform are 0.5 units each) the coordinates of the centroid of the moving platform are given by

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= (1/3)(\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3) \qquad (69) \\ &= \frac{1}{3} \left[\begin{pmatrix} 1 - l_1c_1 \\ 0 \\ l_1s_1 \end{pmatrix} + \begin{pmatrix} (-1/2)(1 - l_2c_2) \\ (\sqrt{3}/2)(1 - l_2c_2) \\ l_2s_2 \end{pmatrix} + \begin{pmatrix} (-1/2)(1 - l_3c_3) \\ (-\sqrt{3}/2)(1 - l_3c_3) \\ l_3s_3 \end{pmatrix} \right] \end{aligned}$$

and the velocity of the center of the top moving platform is given by

$$\mathbf{v} = \frac{1}{3} \left[\begin{pmatrix} l_1s_1 \\ 0 \\ l_1c_1 \end{pmatrix} \dot{\theta}_1 + \begin{pmatrix} (-1/2)(l_2s_2) \\ (\sqrt{3}/2)(l_2s_2) \\ l_2c_2 \end{pmatrix} \dot{\theta}_2 + \begin{pmatrix} (-1/2)(l_3s_3) \\ (-\sqrt{3}/2)(l_3s_3) \\ l_3c_3 \end{pmatrix} \dot{\theta}_3 \right]$$

$$
\begin{aligned}
&+ \ \frac{1}{3}\left[\begin{pmatrix} -c_1 \\ 0 \\ s_1 \end{pmatrix}\dot{l}_1 + \begin{pmatrix} (1/2)(c_2) \\ (-\sqrt{3}/2)(c_2) \\ s_2 \end{pmatrix}\dot{l}_2 + \begin{pmatrix} (1/2)(c_3) \\ (\sqrt{3}/2)(c_3) \\ s_3 \end{pmatrix}\dot{l}_3\right] \\
&= \ [J_v^*]\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} + [J_v]\begin{pmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{l}_3 \end{pmatrix} \qquad (70)
\end{aligned}
$$

where $[J_v^*]$ and $[J_v]$ are 3×3 matrices obtained from the coefficients of $\dot{\theta}_i, i = 1,2,3$ and $\dot{l}_i, i = 1,2,3$ respectively. Using equation (68) in equation (70) we get

$$
\mathbf{v} = ([J_v] - [J_v^*][K^*]^{-1}[K])(\dot{l}_1, \dot{l}_2, \dot{l}_3)^T = \sum_{i=1}^{3} \alpha_i \dot{l}_i \qquad (71)
$$

The angular velocity of the platform can be obtained by observing that

$$
\begin{aligned}
\dot{\mathbf{P}}_2 - \dot{\mathbf{P}}_1 &= \omega \times (\mathbf{P}_2 - \mathbf{P}_1) \\
\dot{\mathbf{P}}_3 - \dot{\mathbf{P}}_1 &= \omega \times (\mathbf{P}_3 - \mathbf{P}_1)
\end{aligned}
$$

and

$$
(\dot{\mathbf{P}}_3 - \dot{\mathbf{P}}_1) \cdot \mathbf{P}_1 = \omega \cdot (\mathbf{P}_3 \times \mathbf{P}_1)
$$

From the above equations, we get

$$
\boldsymbol{\omega} = \frac{(\mathbf{P}_3 \times \mathbf{P}_1) \times (\dot{\mathbf{P}}_2 - \dot{\mathbf{P}}_1) + ((\dot{\mathbf{P}}_3 - \dot{\mathbf{P}}_1) \cdot \mathbf{P}_1)(\mathbf{P}_2 - \mathbf{P}_1)}{\mathbf{P}_1 \times \mathbf{P}_2 \cdot \mathbf{P}_3}
$$

The above can be written in compact form as

$$
\boldsymbol{\omega} \ = \ [J_\omega^*]\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} + [J_\omega]\begin{pmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{l}_3 \end{pmatrix} \qquad (72)
$$

where $[J_\omega^*]$ and $[J_\omega]$ are 3×3 matrices obtained from the coefficients of $\dot{\theta}_i,\ i = 1,2,3,$ and $\dot{l}_i, i = 1,2,3$ respectively. Using equation (68) in equation (72), we get

$$
\boldsymbol{\omega} = ([J_\omega] - [J_\omega^*][K^*]^{-1}[K])(\dot{l}_1, \dot{l}_2, \dot{l}_3)^T = \sum_{i=1}^{3} \beta_i \dot{l}_i \qquad (73)
$$

The dual velocity vector can be written as

$$
\hat{\mathcal{V}} = \sum_{i=1}^{3}(\beta_i + \epsilon\alpha_i)\dot{l}_i = \sum_{i=1}^{3} \hat{\mathcal{S}}_i \dot{l}_i \qquad (74)
$$

Following the mathematical formulation, we can now calculate the dual metric coefficients and dual Christoffel symbols. At a generic point $l_1 = 0.5$ m, $l_2 = 1.0$ m, $l_3 = 2.0$ m, and corresponding $\theta_1 = 0.4000$ rad, $\theta_2 = 0.7535$ rad and $\theta_3 = 0.2402$ rad, the dual metric coefficients are

$$
\begin{aligned}
\hat{g}_{11} &= 2.84834 - \epsilon(0.09046) \\
\hat{g}_{12} = \hat{g}_{21} &= 0.38167 - \epsilon(0.94201) \\
\hat{g}_{13} = \hat{g}_{31} &= -5.76044 - \epsilon(1.40285) \\
\hat{g}_{22} &= 0.72386 + \epsilon(0.43229) \\
\hat{g}_{23} = \hat{g}_{32} &= -2.70705 + \epsilon(2.28721) \\
\hat{g}_{33} &= 17.21660 - \epsilon(3.02947)
\end{aligned}
$$

The dual eigenvalues of $[\hat{g}]$ are given by

$$
\begin{aligned}
\hat{\lambda}_1 &= 19.62130 + \epsilon(-2.48751) \\
\hat{\lambda}_2 &= 1.16742 + \epsilon(-0.20012) \\
\hat{\lambda}_3 &= 0 + \epsilon(0)
\end{aligned}
$$

and the two principal pitches are given by

$$
\begin{aligned}
h_1^* &= -0.06339 \\
h_2^* &= -0.08572 \qquad (75)
\end{aligned}
$$

It can be shown that at other generic points, the dual eigenvalues, $\hat{\lambda}_i,\ i = 1, 2$, are different implying that the pitches are different at different configuration of this parallel manipulator. It may be noted, however, that one of the dual eigenvalue is always zero and the third principal pitch h_3^* is undefined (or infinite) at all configurations. This fact can be resolved as follows: the three degrees-of-freedom of the platform is partitioned into two angular degrees-of-freedom and one pure translatory motion. Intuitively this is clear since the rotary joint axes in the base are in a plane and the top platform can be made to translate parallel to the **Z** axis without any angular motion by changing the leg lengths. One can also show mathematically that one of the principal velocities is a pure dual. In particular, at the configuration mentioned above, the principal velocities obtained from equation(34) are

$$\begin{aligned}
\hat{\mathcal{V}}_1 &= (-1.71201, -4.04936, 0.54134)^T + \epsilon(0.60671, 0.30574, 1.90821)^T \\
\hat{\mathcal{V}}_2 &= (0.34667, -0.00977, 1.02330)^T + \epsilon(0.38057, -0.34252, -0.22998)^T \\
\hat{\mathcal{V}}_3 &= (0, 0, 0)^T + \epsilon(0, 0, 1.21575)^T
\end{aligned}$$

The dual part of $\hat{\mathcal{V}}_3$ is always in the $\mathbf{Z}$ direction, for all points attached to the platform frame, in any configuration. One can also compute the 18 dual Christoffel symbols. In this case, they are non-zero. For example, $\hat{\Gamma}_{111}$ at the given configuration is $-17.02180 - \epsilon(1.62009)$. The geodesic motion for this case cannot be solved easily, as in the case of the 2R manipulator, since it involves solution of non-linear differential equations.

5 Conclusion

In this paper we have given a new geometric characterization of the differential kinematics of two- and three-degree-of-freedom rigid body motion. This new characterization has been obtained by using dual numbers and vectors, expressing the linear and angular velocities of a rigid body by means of a dual vector, defining an inner product between two dual vectors as a dual number, and by using concepts from differential geometry. The main result of the paper is that the tip of a dual velocity vector of a rigid body lies on a dual ellipse or a dual ellipsoid for a two- and three-degree-of-freedom motion respectively, and the maximum and minimum values of the dual velocity vectors are the eigenvalues of a positive, semi-definite dual matrix of inner products. In this fashion velocity distribution in a rigid body motion can be studied algebraically in terms of eigenvalues or geometrically from the ellipse and the dual ellipsoid. We have also shown that the second-order properties of rigid body motions can be studied in a similar fashion. In this regard, we have presented the notion of the geodesic motion of a rigid body. The theoretical results have been illustrated with the help of spatial 2R and a three-degree-of-freedom parallel manipulator.

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