A note on the diagonalizability and the Jordan form of the 4×4 homogeneous transformation matrix

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Abstract

The 4×4 homogeneous transformation matrix is extensively used for representing rigid body displacement in 3D space and has been extensively used in the analysis of mechanisms, serial and parallel manipulators, and in the field of geometric modeling and CAD. The properties of the transformation matrix are very well known. One of the well known properties is that a general 4×4 homogeneous transformation matrix cannot be diagonalized, and at best can be reduced to a Jordan form . In this note, we show that the 4×4 homogeneous transformation matrix *can* be diagonalized if and only if displacement along the screw axis is zero. For the general transformation with non-zero displacement along the axis, we present an explicit expression for the fourth basis vector of the Jordan basis. We also present a variant of the Jordan form which contains the motion variables along and about the screw axis and the corresponding basis vectors which contains the information only about the screw axis and its location. We present a novel expression for a point on the screw axis closest to the origin, which is then used to form a simple choice of basis for different forms. Finally the theoretical results are illustrated with a numerical example.

1 Introduction

The 4×4 homogenous transformation matrix, denoted by $[\boldsymbol{T}]$, represents a general displacement of a rigid body in 3D space. It has the form

$$[\boldsymbol{T}] = \begin{bmatrix} [\boldsymbol{R}] & \boldsymbol{d} \\ \boldsymbol{0} & 1 \end{bmatrix} \tag{1}$$

where the 3×3 orthogonal matrix $[\boldsymbol{R}]$, with determinant$([\boldsymbol{R}]) = +1$, represents rotation of the rigid body, $\boldsymbol{0} = [\ 0\ 0\ 0\ ]$ and $\boldsymbol{d} \in \mathbb{R}^{\boldsymbol{3}}$ represents the translation of the rigid body.[1] It is well known in literature (see, for example, McCarthy 1990) that the eigenvalues of $[\boldsymbol{T}]$ are $e^{\imath\phi}$, $e^{-\imath\phi}$, 1 and 1, where ϕ is the screw rotation. Evidently the algebraic multiplicity of the real eigenvalue, $\lambda = 1$, is 2. It is also known that $(\boldsymbol{k}, 0)$, $(\boldsymbol{x_1}, 0)$ and $(\boldsymbol{x_2}, 0) \in \mathbb{C}^{\boldsymbol{4}}$ are eigenvectors of $[\boldsymbol{T}]$ where the 3×1 vectors $\boldsymbol{k}$, $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are the eigenvectors of the orthogonal matrix $[\boldsymbol{R}]$, corresponding

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[1] Unless stated otherwise, we assume $[\boldsymbol{R}]$ to be nontrivial, i.e $[\boldsymbol{R}] \neq [\boldsymbol{I_3}]$.

to the eigenvalues $\lambda = 1$, $e^{\imath\phi}$ and $e^{-\imath\phi}$, respectively[2]. For a general transformation matrix, $[\boldsymbol{T}]$, we cannot find a *fourth* eigenvector for the repeated eigenvalue $\lambda = 1$, i.e., the geometric multiplicity for $\lambda = 1$ is 1. As a consequence, the general $[\boldsymbol{T}]$ matrix *cannot* be diagonalized and can, at best, be reduced to a Jordan form (see, for example, McCarthy 1990).

In this note, we present the necessary and sufficient condition for the 4×4 homogenous transformation matrix, $[\boldsymbol{T}]$, to possess geometric multiplicity of 2 for $\lambda = 1$, and as a consequence, the necessary and sufficient conditions for $[\boldsymbol{T}]$ to be diagonalizable. Although some of the results are well known in literature, in this note, the stress is on rigorous linear algebraic treatment and obtaining explicit formulas and proofs. The diagonalizability of homogeneous transformation matrix is discussed in Section 2. In Section 3, we first present explicit expressions for the Jordan basis of general non-diagonalizable $[\boldsymbol{T}]$ and then a novel geometric derivation of the vector indicating the screw axis. The derived formula is then used to give a simple choice of Jordan and eigenbasis for non-diagonalizable and diagonalizable $[\boldsymbol{T}]$, respectively. In Section 3, we also present a variant of the Jordan form and its basis with the modified Jordan form containing the motion variables about and along the screw axis and the corresponding basis containing the information about the screw axis and its location. In Section 4, we present an example to illustrate the theoretical results.

2 Diagonalizability of $[\boldsymbol{T}]$

A square matrix $[\boldsymbol{A}]$ in a linear transformation $[\boldsymbol{A}]\boldsymbol{X} = \boldsymbol{Y}$ can be diagonalized if the eigenvalues of $[\boldsymbol{A}]$ are distinct. In case of repeated eigenvalues, $[\boldsymbol{A}]$ can still be diagonalized if the algebraic multiplicity of the repeated eigenvalues are equal to their geometric multiplicity (Hoffman and Kunze 1971). For the 4×4 homogeneous transformation matrix, $[\boldsymbol{T}]$, the algebraic multiplicity of $\lambda = 1$ is 2 and $[\boldsymbol{T}]$ can be diagonalized if and only if the geometric multiplicity of $\lambda = 1$ is 2. The matrix $[\boldsymbol{T}]$ will have geometric multiplicity of 2 if nullity($[\boldsymbol{T} - \boldsymbol{I_4}]$) $= 2$ or if there exists two linearly independent vectors which span the null space of $[\boldsymbol{T} - \boldsymbol{I_4}]$ – in such a case those two linearly independent vectors, along with $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, will form an eigenbasis for $\mathbb{C}^4$ and with respect to this basis $[\boldsymbol{T}]$ will become diagonalized. One vector which is always in the null space is $(\boldsymbol{k}, 0)$ where the 3×1 vector $\boldsymbol{k}$ is the eigenvector of $[\boldsymbol{R}]$ for the eigenvalue $\lambda = 1$. Let the other, if it exists, be denoted by $\tilde{\boldsymbol{t}}$. It has been reasoned in McCarthy(1990) that the last component of $\tilde{\boldsymbol{t}}$ cannot be zero. Hence, $\tilde{\boldsymbol{t}}$ should have the form [3]

$$\tilde{\boldsymbol{t}} = (\boldsymbol{u}, 0) + \gamma(0, 0, 0, 1), \qquad \gamma \neq 0 \tag{2}$$

However, $\tilde{\boldsymbol{t}}$ is required to be eigenvector with $\lambda = 1$, hence

$$\begin{aligned} [\boldsymbol{T} - \boldsymbol{I_4}]\tilde{\boldsymbol{t}} &= \tilde{\boldsymbol{0}} \\ \Rightarrow ([\boldsymbol{R} - \boldsymbol{I_3}]\boldsymbol{u}, 0) + (\gamma\boldsymbol{d}, 0) &= \tilde{\boldsymbol{0}} \\ \Rightarrow [\boldsymbol{R} - \boldsymbol{I_3}]\boldsymbol{u} &= -\gamma\boldsymbol{d} \end{aligned} \tag{3}$$

where $\boldsymbol{I_k}$ indicates the $k \times k$ identity matrix.

[2]It may be noted that since $[\boldsymbol{R}]$ is real, the eigenvectors $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are related by $\boldsymbol{x_2} = a\bar{\boldsymbol{x}}_{\boldsymbol{1}}$, for some $a \in \mathbb{C}$, where $\bar{x}_1$ is complex conjugate of $\boldsymbol{x_1}$.

[3]Four-dimensional vectors are indicated in boldface lower-case and by using a tilde

Since $[\boldsymbol{R}-\boldsymbol{I_3}]$ is singular, the solution for $\boldsymbol{u}$ does not exist for a general $\boldsymbol{d}$ (McCarthy 1990). A solution for $\boldsymbol{u}$ can exist in eq. (3) only if $\boldsymbol{d}$ is in the range space of $[\boldsymbol{R}-\boldsymbol{I_3}]$ (Hoffman and Kunze 1971). The latter holds even if $\boldsymbol{d}$ is a zero vector.

Let $\boldsymbol{U}$ be the subspace perpendicular to $\boldsymbol{k}$ (geometrically $\boldsymbol{U}$ represents a plane perpendicular to $\boldsymbol{k}$ and passing through the origin). For a given nontrivial $[\boldsymbol{R}]$, one can find vectors $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ such that $\boldsymbol{c_1}\cdot\boldsymbol{c_2}=\boldsymbol{c_1}\cdot\boldsymbol{k}=\boldsymbol{c_2}\cdot\boldsymbol{k}=0$, $|\boldsymbol{c_1}|=|\boldsymbol{c_2}|$ and

$$[\boldsymbol{R}]\boldsymbol{c_1}=\boldsymbol{c_1}\cos\phi+\boldsymbol{c_2}\sin\phi \quad (4)$$

$$[\boldsymbol{R}]\boldsymbol{c_2}=-\boldsymbol{c_1}\sin\phi+\boldsymbol{c_2}\cos\phi \quad (5)$$

Evidently $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ qualify as a basis of $\boldsymbol{U}$. It can also be shown that $\{\boldsymbol{k},\boldsymbol{c_1},\boldsymbol{c_2}\}$ represents a basis of $\mathbb{R}^{\boldsymbol{3}}$. We prove that $\boldsymbol{U}$ is the range space of $[\boldsymbol{R}-\boldsymbol{I_3}]$:
Consider an arbitrary vector $\boldsymbol{b}\in\mathbb{R}^{\boldsymbol{3}}$ given by

$$\boldsymbol{b}=\alpha_1\boldsymbol{k}+\alpha_2\boldsymbol{c_1}+\alpha_3\boldsymbol{c_2}$$

where α_1, α_2 and $\alpha_3\in\mathbb{R}$. Then from eqs. (4) and (5),

$$[\boldsymbol{R}-\boldsymbol{I_3}]\boldsymbol{b}=(\alpha_2(\cos\phi-1)-\alpha_3\sin\phi)\boldsymbol{c_1}+(\alpha_2\sin\phi+\alpha_3(\cos\phi-1))\boldsymbol{c_2}\ \in\ \boldsymbol{U}$$

and hence for any $\boldsymbol{b}\in\mathbb{R}^{\boldsymbol{3}}$, we get $[\boldsymbol{R}-\boldsymbol{I_3}]\boldsymbol{b}\in\boldsymbol{U}$.
Next consider a vector $\boldsymbol{d}\in\boldsymbol{U}$ given by

$$\boldsymbol{d}=\beta_1\boldsymbol{c_1}+\beta_2\boldsymbol{c_2}$$

where β_1 and $\beta_2\in\mathbb{R}$, then there exists a vector $\boldsymbol{f}$ given by

$$\boldsymbol{f}=\frac{(\beta_1(\cos\phi-1)+\beta_2\sin\phi)}{2(1-\cos\phi)}\boldsymbol{c_1}+\frac{(-\beta_1\sin\phi+\beta_2(\cos\phi-1))}{2(1-\cos\phi)}\boldsymbol{c_2}+\alpha\boldsymbol{k}$$

where $\alpha\in\mathbb{R}$. It can be shown that : $[\boldsymbol{R}-\boldsymbol{I_3}]\boldsymbol{f}=\boldsymbol{d}\quad\forall\alpha$. Hence, for any $\boldsymbol{d}\in\boldsymbol{U}$ there exists $\boldsymbol{f}\in\mathbb{R}^{\boldsymbol{3}}$ such that $[\boldsymbol{R}-\boldsymbol{I_3}]\boldsymbol{f}=\boldsymbol{d}$. It may be noted that $\boldsymbol{f}$ exists only if $\phi\neq 2n\pi$ with $n\in\mathbb{Z}$, which brings out the importance of the assumption that $[\boldsymbol{R}]$ is nontrivial. This proves that $\boldsymbol{U}$ is the range space of $[\boldsymbol{R}-\boldsymbol{I_3}]$.

Going back to eq. (3), for the existence of $\boldsymbol{u}$ (and by consequence the fourth eigenvector $\tilde{\boldsymbol{t}}$ of $[\boldsymbol{T}]$), $\boldsymbol{d}$ must lie in $\boldsymbol{U}$, and this implies that $\boldsymbol{d}$, which in general can be any vector in $\mathbb{R}^{\boldsymbol{3}}$, is constrained to be of the form $\boldsymbol{d}=\beta_1\boldsymbol{c_1}+\beta_2\boldsymbol{c_2}$, with $\beta_1,\beta_2\in\mathbb{R}$. For such a $\boldsymbol{d}$, we have $\hat{\boldsymbol{k}}\cdot\boldsymbol{d}=\beta_1\hat{\boldsymbol{k}}\cdot\boldsymbol{c_1}+\beta_2\hat{\boldsymbol{k}}\cdot\boldsymbol{c_2}=0$, where $\hat{\boldsymbol{k}}=\boldsymbol{k}/|\boldsymbol{k}|$. However, $\hat{\boldsymbol{k}}\cdot\boldsymbol{d}=\delta$ is the magnitude of the displacement along the screw axis, and hence diagonalizability of $[\boldsymbol{T}]$ implies that $\delta=0$.

The converse of the above is also true – if $\delta=\hat{\boldsymbol{k}}\cdot\boldsymbol{d}=0$, then by definition of $\boldsymbol{U}$, the vector $\boldsymbol{d}$ lies in $\boldsymbol{U}$ and hence a solution for $\boldsymbol{u}$ exists in eq. (3). A vector $(\boldsymbol{u},\gamma)$ can then be constructed, and we have

$$[\boldsymbol{T}-\boldsymbol{I_4}](\boldsymbol{u},\gamma)=([\boldsymbol{R}-\boldsymbol{I_3}]\boldsymbol{u}+\gamma\boldsymbol{d},0)=(\boldsymbol{0},0) \quad (6)$$

which implies that $(\boldsymbol{u},\gamma)$ is an eigenvector of $[\boldsymbol{T}]$ for the eigenvalue $\lambda=1$.
Since $\gamma\neq 0$, the vectors $(\boldsymbol{x_1},0),(\boldsymbol{x_2},0),(\boldsymbol{k},0)$ and $(\boldsymbol{u},\gamma)$ are linearly independent. They are four in number (same as the dimension of $\mathbb{C}^{\boldsymbol{4}}$), and hence they form an eigenbasis for $\mathbb{C}^{\boldsymbol{4}}$. This implies that $[\boldsymbol{T}]$ can be diagonalized.

Therefore we have :

Result: A 4×4 homogeneous transformation with nontrivial rotation is diagonalizable if and only if the displacement along the screw axis is zero.

Remark: 1) The vectors $(\boldsymbol{u}, \gamma)$ and $(\boldsymbol{k}, 0)$ span a space of dimension 2 in $\mathbb{R}^{\mathbf{4}}$. We denote this space by $\boldsymbol{E_1}$. Every vector in $\boldsymbol{E_1}$ is an eigenvector for $\lambda = 1$ and hence it is an eigenspace for $[\boldsymbol{T}]$.

Remark: 2) If $\boldsymbol{u}$ satisfies eq. (3), then $\boldsymbol{u} + \alpha\boldsymbol{k}$ also satisfies eq. (3). In addition, if $\gamma \neq 0$ is replaced by $\beta \neq 0$ in eq. (3), then $\beta\boldsymbol{u}/\gamma + \alpha\boldsymbol{k}$ also satisfies the eq. (3), for any value of α. However, the vector $(\beta\boldsymbol{u}/\gamma + \alpha\boldsymbol{k}, \beta)$ lies in $\boldsymbol{E_1}$. Hence non uniqueness of $\boldsymbol{u}$ and arbitrariness of γ in the proof of the result is accounted by many possible choices of eigenvectors in $\boldsymbol{E_1}$.

Remark: 3) The eigenspace $\boldsymbol{E_1}$ is geometrically significant since the coordinates of a vector at the intersection of hyperplane $x_4 = 1$ and $\boldsymbol{E_1}$, are the homogeneous coordinates of a point on the screw axis. [4]

3 A basis for the Jordan form

When the displacement along the screw axis, δ, is non zero, $[\boldsymbol{T}]$ can't be diagonalized, but it can be reduced to a Jordan form. We know from linear algebra that to obtain the Jordan basis corresponding to the repeated eigenvalue $\lambda = 1$, we need to consider the nullspace of $[\boldsymbol{T} - \boldsymbol{I_4}]^{\mathbf{2}}$. To obtain the basis, we follow the treatment in McCarthy(1990) in more detail.

Let $\boldsymbol{E_2}$ denote the nullspace of $[\boldsymbol{T} - \boldsymbol{I_4}]^{\mathbf{2}}$. Evidently, $(\boldsymbol{k}, 0)$ lies in $\boldsymbol{E_2}$. Consider a vector $\boldsymbol{v}$ which satisfies

$$[\boldsymbol{R} - \boldsymbol{I_3}]\boldsymbol{v} = -\gamma\boldsymbol{d_p}, \quad \gamma \neq 0 \tag{7}$$

where $\boldsymbol{d_p}$ is the projection of $\boldsymbol{d}$ on the plane perpendicular to $\boldsymbol{k}$ and

$$\boldsymbol{d} = \boldsymbol{d_p} + \delta\hat{\boldsymbol{k}} \tag{8}$$

In such a case, $(\boldsymbol{v}, \gamma)$ also lies in $\boldsymbol{E_2}$, since

$$\begin{aligned} [\boldsymbol{T}](\boldsymbol{v}, \gamma) &= ([\boldsymbol{R}]\boldsymbol{v} + \gamma\boldsymbol{d}, \gamma) \\ &= (\boldsymbol{v}, \gamma) + \gamma(\delta\hat{\boldsymbol{k}}, 0) \end{aligned} \tag{9}$$

where the last step is obtained by using eqs. (7) and (8). From eq. (9), we can get $[\boldsymbol{T} - \boldsymbol{I_4}](\boldsymbol{v}, \gamma) = \gamma\delta(\hat{\boldsymbol{k}}, 0)$, and noting that $(\hat{\boldsymbol{k}}, 0)$ is in the nullspace of $[\boldsymbol{T} - \boldsymbol{I_4}]$, we get

$$[\boldsymbol{T} - \boldsymbol{I_4}]^{\mathbf{2}}(\boldsymbol{v}, \gamma) = (\mathbf{0}, 0) \tag{10}$$

It may be noted that $\boldsymbol{v}$ exists, though not unique, since $\boldsymbol{d_p}$ is in the range space of $[\boldsymbol{R} - \boldsymbol{I_3}]$. Also, since $\gamma \neq 0$, $(\boldsymbol{k}, 0)$, $(\boldsymbol{v}, \gamma)$, $(\boldsymbol{x_1}, 0)$ and $(\boldsymbol{x_2}, 0)$ are linearly independent, they span $\mathbb{C}^{\mathbf{4}}$, and hence form a basis of $\mathbb{C}^{\mathbf{4}}$.

[4] The symbols x_1, x_2, x_3 and x_4 are the coordinates of a vector in $\mathbb{R}^{\mathbf{4}}$, with respect to standard ordered basis $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$.

We can also show that $(\boldsymbol{k},0),(\boldsymbol{v},\gamma)$ span $\boldsymbol{E_2}$. Consider any vector $\tilde{\boldsymbol{t}} = \alpha_1(\boldsymbol{k},0) + \alpha_2(\boldsymbol{v},\gamma) + \alpha_3(\boldsymbol{x_1},0) + \alpha_4(\boldsymbol{x_2},0)$. If $\tilde{\boldsymbol{t}} \in \boldsymbol{E_2}$, then by definition of $\boldsymbol{E_2}$, we have $[\boldsymbol{T}-\boldsymbol{I_4}]^2\tilde{\boldsymbol{t}} = 0$. Noting that two of the basis vectors $(\boldsymbol{k},0),(\boldsymbol{v},\gamma)$ belong to $\boldsymbol{E_2}$, we get

$$[\boldsymbol{T}-\boldsymbol{I_4}]^2\tilde{\boldsymbol{t}} = \alpha_3(e^{i\phi}-1)^2(\boldsymbol{x_1},0) + \alpha_4(e^{-i\phi}-1)^2(\boldsymbol{x_2},0) = \tilde{\boldsymbol{0}}$$

Since $(\boldsymbol{x_1},0)$ and $(\boldsymbol{x_2},0)$ are linearly independent and for $\theta \neq 2n\pi$ or $(e^{\pm i\phi}-1) \neq 0$, we get $\alpha_3 = \alpha_4 = 0$. Hence $\tilde{\boldsymbol{t}} \in \boldsymbol{E_2}$ implies $\tilde{\boldsymbol{t}}$ is in the span of $(\boldsymbol{k},0)$ and $(\boldsymbol{v},\gamma)$, and therefore span of $(\boldsymbol{k},0)$ and $(\boldsymbol{v},\gamma)$ is $\boldsymbol{E_2}$. It is also known that $\boldsymbol{E_2}$ is an invariant subspace and vectors lying at the intersection of $\boldsymbol{E_2}$ and hyperplane $x_4 = 1$, give the homogenous coordinates of points on the screw axis (McCarthy 1990).

To obtain a Jordan basis, consider the vectors $\tilde{\boldsymbol{f}}_{\boldsymbol{1}}$, $\tilde{\boldsymbol{f}}_{\boldsymbol{2}}$, $\tilde{\boldsymbol{f}}_{\boldsymbol{3}}$ and $\tilde{\boldsymbol{f}}_{\boldsymbol{4}}$ as ordered basis of $\mathbb{C}^{\boldsymbol{4}}$, such that the full-rank 4×4 change of basis matrix $[\boldsymbol{F}] = [\tilde{\boldsymbol{f}}_{\boldsymbol{1}}\ \tilde{\boldsymbol{f}}_{\boldsymbol{2}}\ \tilde{\boldsymbol{f}}_{\boldsymbol{3}}\ \tilde{\boldsymbol{f}}_{\boldsymbol{4}}]$ satisfies $[\boldsymbol{T}][\boldsymbol{F}] = [\boldsymbol{F}][\boldsymbol{J}]$, where $[\boldsymbol{J}]$ is the following Jordan Form.

$$\begin{bmatrix} e^{i\phi} & 0 & 0 & 0 \\ 0 & e^{-i\phi} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The condition that $[\boldsymbol{T}][\boldsymbol{F}] = [\boldsymbol{F}][\boldsymbol{J}]$ is equivalent the following equations:

$$\begin{aligned} [\boldsymbol{T}]\tilde{\boldsymbol{f}}_{\boldsymbol{1}} &= e^{i\phi}\tilde{\boldsymbol{f}}_{\boldsymbol{1}} \\ [\boldsymbol{T}]\tilde{\boldsymbol{f}}_{\boldsymbol{2}} &= e^{-i\phi}\tilde{\boldsymbol{f}}_{\boldsymbol{2}} \\ [\boldsymbol{T}]\tilde{\boldsymbol{f}}_{\boldsymbol{3}} &= \tilde{\boldsymbol{f}}_{\boldsymbol{3}} \\ [\boldsymbol{T}]\tilde{\boldsymbol{f}}_{\boldsymbol{4}} &= \tilde{\boldsymbol{f}}_{\boldsymbol{4}} + \tilde{\boldsymbol{f}}_{\boldsymbol{3}} \end{aligned}$$

For $\delta \neq 0$, $\tilde{\boldsymbol{f}}_{\boldsymbol{1}} = (\boldsymbol{x_1},0)$,$\tilde{\boldsymbol{f}}_{\boldsymbol{2}} = (\boldsymbol{x_2},\boldsymbol{0})$, $\tilde{\boldsymbol{f}}_{\boldsymbol{3}} = (\boldsymbol{k},0)$ are obviously eigenvectors. Then

$$[\boldsymbol{T}]\tilde{\boldsymbol{f}}_{\boldsymbol{4}} = \tilde{\boldsymbol{f}}_{\boldsymbol{4}} + (\boldsymbol{k},0) = \tilde{\boldsymbol{f}}_{\boldsymbol{4}} + (|k|\hat{\boldsymbol{k}},0) \tag{11}$$

Clearly $\tilde{\boldsymbol{f}}_{\boldsymbol{4}}$ must lie in $\boldsymbol{E_2}$ but cannot be a scalar multiple of $(\boldsymbol{k},0)$ since it will then become linearly dependent on $\tilde{\boldsymbol{f}}_{\boldsymbol{3}}$. Hence $\tilde{\boldsymbol{f}}_{\boldsymbol{4}}$ should have the form

$$\tilde{\boldsymbol{f}}_{\boldsymbol{4}} = \alpha(\boldsymbol{k},0) + \beta(\boldsymbol{v},\gamma) \qquad \text{where } \beta \neq 0$$

Using eq. (9) and eq. (11), we get the following condition on $\beta\gamma$:

$$\beta\gamma\delta = |k| \qquad \text{or} \qquad \beta\gamma = \frac{|k|}{\delta}$$

Hence, requiring $\tilde{\boldsymbol{f}}_{\boldsymbol{4}}$ to be the basis for the Jordan form implies that it lies in the intersection of $\boldsymbol{E_2}$ and the hyperplane $x_4 = \frac{|\boldsymbol{k}|}{\delta}$ i.e., it should be of the form

$$\tilde{\boldsymbol{f}}_{\boldsymbol{4}} = \alpha(\boldsymbol{k},0) + (\boldsymbol{w},\frac{|\boldsymbol{k}|}{\delta}) \qquad \text{where } \boldsymbol{w} \text{ satisfies} \qquad [\boldsymbol{R}-\boldsymbol{I_3}]\boldsymbol{w} = -\frac{|\boldsymbol{k}|}{\delta}\boldsymbol{d_p} \tag{12}$$

and α is arbitrary scalar. One can verify that $\alpha(\boldsymbol{k},0) + (\boldsymbol{w},\frac{|\boldsymbol{k}|}{\delta})$ does satisfy eq. (11). Hence vectors $\tilde{\boldsymbol{f}}_{\boldsymbol{1}}$, $\tilde{\boldsymbol{f}}_{\boldsymbol{2}}$, $\tilde{\boldsymbol{f}}_{\boldsymbol{3}}$ and $\tilde{\boldsymbol{f}}_{\boldsymbol{4}}$, which satisfy above conditions, form the basis, with respect to which $[\boldsymbol{T}](\phi \neq 2n\pi)$ transforms to the Jordan form.

Note: Given $\tilde{\boldsymbol{f_1}}$, $\tilde{\boldsymbol{f_2}}$, $\tilde{\boldsymbol{f_3}}$ and $\tilde{\boldsymbol{f_4}}$, as the basis for the Jordan form, the interpretation of $\tilde{\boldsymbol{f_1}} = (\boldsymbol{x_1}, 0)$, $\tilde{\boldsymbol{f_2}} = (\boldsymbol{x_2}, 0)$ and $\tilde{\boldsymbol{f_3}} = (\boldsymbol{k}, 0)$ are well known. From the fourth vector $\tilde{\boldsymbol{f_4}}$, we can obtain the magnitude of the displacement along the screw axis and the location of the screw axis. They are given by

$$\begin{aligned} \delta &= \frac{|\boldsymbol{k}|}{f_4(4)} \\ \boldsymbol{r_1} &= \frac{\boldsymbol{f_4}}{f_4(4)} \end{aligned}$$

where $f_4(4)$ is the fourth component of $\tilde{\boldsymbol{f_4}}$, $\boldsymbol{r_1}$ is some point on the screw axis and $\boldsymbol{f_4}$ is the vector in $\mathbb{R}^{\boldsymbol{3}}$, formed by omitting the last component of $\tilde{\boldsymbol{f_4}}$.

It is useful to note that had $\tilde{\boldsymbol{f_4}}$ in eq. (12) been chosen as

$$\tilde{\boldsymbol{f_4}} = \alpha(\boldsymbol{k}, 0) + (\boldsymbol{w}, |\boldsymbol{k}|) \qquad \text{where } \boldsymbol{w} \text{ satisfies} \qquad [\boldsymbol{R} - \boldsymbol{I_3}]\boldsymbol{w} = -|\boldsymbol{k}|\boldsymbol{d_p} \tag{13}$$

and $\tilde{\boldsymbol{f_1}} = (\boldsymbol{x_1}, 0)$, $\tilde{\boldsymbol{f_2}} = (\boldsymbol{x_2}, \boldsymbol{0})$, $\tilde{\boldsymbol{f_3}} = (\boldsymbol{k}, 0)$ being same as that for Jordan basis, then under the new basis $\tilde{\boldsymbol{f_1}}$, $\tilde{\boldsymbol{f_2}}$, $\tilde{\boldsymbol{f_3}}$ and $\tilde{\boldsymbol{f_4}}$, the matrix $[\boldsymbol{T}]$, after the similarity transformation, assumes the following canonical form

$$[\boldsymbol{C}](\phi, \delta) = \begin{bmatrix} e^{i\phi} & 0 & 0 & 0 \\ 0 & e^{-i\phi} & 0 & 0 \\ 0 & 0 & 1 & \delta \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

The last column of the matrix in eq. (14) implies that the condition $[\boldsymbol{T}]\tilde{\boldsymbol{f_4}} = \tilde{\boldsymbol{f_4}} + \delta\tilde{\boldsymbol{f_3}}$ should be satisfied and it is verified as below.

$$\begin{aligned} [\boldsymbol{T}]\tilde{\boldsymbol{f_4}} &= [\boldsymbol{T}](\alpha(\boldsymbol{k}, 0) + (\boldsymbol{w}, |\boldsymbol{k}|)) \\ &= \alpha(\boldsymbol{k}, 0) + ([\boldsymbol{R}]\boldsymbol{w} + |\boldsymbol{k}|\boldsymbol{d}, |\boldsymbol{k}|) \\ &= \alpha(\boldsymbol{k}, 0) + (-|\boldsymbol{k}|\boldsymbol{d_p} + \boldsymbol{w} + |\boldsymbol{k}|(\boldsymbol{d_p} + \delta\hat{\boldsymbol{k}}), |\boldsymbol{k}|) \quad \text{(using condition on } \boldsymbol{w} \text{ in eqn(13))} \\ &= \alpha(\boldsymbol{k}, 0) + (\boldsymbol{w}, |\boldsymbol{k}|) + (\delta|\boldsymbol{k}|\hat{\boldsymbol{k}}, 0) = \tilde{\boldsymbol{f_4}} + \delta\tilde{\boldsymbol{f_3}} \end{aligned}$$

This combination of basis, which is a variation of basis for the Jordan form, along with the form $[\boldsymbol{C}](\phi, \delta)$ turns out be be more informative and general. It is important to note that only screw axis information is present in the basis vector (unlike the basis for Jordan form), while both rotation ϕ and axial movement δ is manifested in transformed matrix. Also this form is general in the sense that it encompasses both diagonal and non-diagonal case. Some useful implications of this form and its corresponding basis are presented later in the section.

Of all the points on the screw axis, let us consider the closest point to the origin. Figure 1(a) shows two configurations of a rigid body denoted by $\{A\}$ and $\{B\}$. The 4×4 transformation matrix, ${}^A_B[\boldsymbol{T}]$, denotes the rigid body displacement from $\{A\}$ to $\{B\}$. The associated screw axis $\mathcal{S}$ is also shown in figure 1(a) and located by the vector $\boldsymbol{r_0}$, the position vector of the closest point S on the screw axis with respect to $\{A\}$. Since $\boldsymbol{r_0}$ is assumed to be the closest point, it is perpendicular to $\mathcal{S}$ and lies in the plane perpendicular to $\hat{\boldsymbol{k}}$ and passing through O_A. We call this plane $\boldsymbol{U}$, shown by partial hatching in figure 1(a) and by uniform shading in the figure 1(b). Let a point fixed

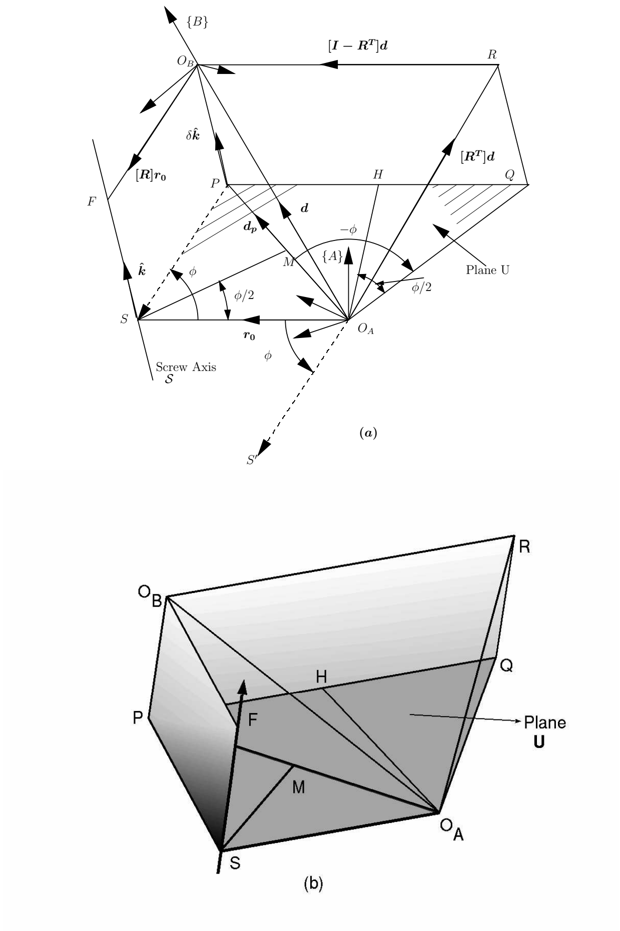

Figure 1: Geometrical construction to derive $\boldsymbol{r_0}$

to the rigid body and initially coincident with S be called S'. The complete displacement of the rigid body can be decomposed into $i)$ a rotation of angle ϕ about an axis parallel to $\hat{\boldsymbol{k}}$ and passing through O_A, $ii)$ a translation by $\boldsymbol{d_p}$, and $iii)$ a translation of δ along $\hat{\boldsymbol{k}}$. Due to the first rotation we get $\overrightarrow{O_AS'} = [\boldsymbol{R}]\boldsymbol{r_0}$ and it still lies in $\boldsymbol{U}$. Due to translation by $\boldsymbol{d_p}$, S' coincides back with S since S was chosen to lie on the screw axis $\mathcal{S}$. It can be observed that $\overrightarrow{O_AS} = \boldsymbol{r_0}$, $\overrightarrow{O_AP} = \boldsymbol{d_p}$ and $\overrightarrow{PS} = [\boldsymbol{R}]\boldsymbol{r_0}$, all lie in the plane $\boldsymbol{U}$ and form the sides of the triangle O_ASP. The latter is isosceles with $SO_A = PS$, apex angle $\angle O_ASP = \phi$ and SM as the bisector of the apex angle. Finally, the translation $\delta\hat{\boldsymbol{k}}$ in $iii)$ translates S' along the screw axis to bring it to F.

Moreover, the vector $\overrightarrow{O_AO_B} = \overrightarrow{O_AP} + \overrightarrow{PO_B}$, i.e. $\boldsymbol{d} = \boldsymbol{d_p} + \delta\hat{\boldsymbol{k}}$. Multiplication by $\boldsymbol{R^T}$ implies a rotation by $-\phi$, about an axis along $\hat{\boldsymbol{k}}$ and passing through O_A. The vectors $[\boldsymbol{R^T}]\boldsymbol{d}$ and $[\boldsymbol{R^T}]\boldsymbol{d_p}$ are indicated by $\overrightarrow{O_AR}$ and $\overrightarrow{O_AQ}$, respectively. Furthermore we have $\boldsymbol{d} = \overrightarrow{O_AO_B} = \overrightarrow{O_AP} + \overrightarrow{PO_B}$ where $\overrightarrow{O_AP} = \boldsymbol{d_p}$ and $\overrightarrow{PO_B} = \delta\hat{\boldsymbol{k}}$. Likewise, we have $[\boldsymbol{R^T}]\boldsymbol{d} = \overrightarrow{O_AR} = \overrightarrow{O_AQ} + \overrightarrow{QR}$ where $\overrightarrow{O_AQ} = [\boldsymbol{R^T}]\boldsymbol{d_p}$ and $\overrightarrow{QR} = \delta\hat{\boldsymbol{k}}$. For $\boldsymbol{d} = \boldsymbol{d_p} + \delta\hat{\boldsymbol{k}}$, we get

$$[\boldsymbol{I_3} - \boldsymbol{R^T}]\boldsymbol{d} = \overrightarrow{O_AO_B} - \overrightarrow{O_AR} = \overrightarrow{O_AP} - \overrightarrow{O_AQ} = \overrightarrow{QP}$$

We again note that the three vectors, $\overrightarrow{O_AQ}$, $\overrightarrow{QP}$, $\overrightarrow{O_AP}$, lie in the plane $\boldsymbol{U}$ and the triangle O_AQP is similar to triangle SO_AP with O_AH as the bisector of the apex angle. From the geometry of two triangles, we get

$$O_AS \parallel QP \qquad \text{and} \quad \frac{O_AS}{QP} = \frac{1}{4\sin^2\frac{\phi}{2}} = \frac{1}{2(1-\cos\phi)}$$

The above implies that $i)$ the vectors $\overrightarrow{O_AS} = \boldsymbol{r_0}$ and $\overrightarrow{QP} = [\boldsymbol{I_3} - \boldsymbol{R^T}]\boldsymbol{d}$ have the same direction and that $ii)$ the ratio of their magnitudes is as given above. Hence we get

$$\boldsymbol{r_0} = \frac{[\boldsymbol{I_3} - \boldsymbol{R^T}]\boldsymbol{d}}{2(1-\cos\phi)} = \frac{[\boldsymbol{I_3} - \boldsymbol{R^T}]\boldsymbol{d}}{3 - \text{trace}([\boldsymbol{R}])} \tag{15}$$

where we have used the equality $\text{trace}([\boldsymbol{R}]) = 1 + 2\cos\phi$.

The formula for $\boldsymbol{r_0}$ can also be justified as follows: first, for $\boldsymbol{r_0}$ to be a point of the screw axis, $\boldsymbol{r_0}$ must satisfy $[\boldsymbol{R} - \boldsymbol{I_3}]\boldsymbol{r_0} = -\boldsymbol{d_p}$. Since $\boldsymbol{d_p}$ lies in a subspace perpendicular to $\hat{\boldsymbol{k}}$, we can write

$$\boldsymbol{d_p} = \alpha_1\boldsymbol{c_1} + \alpha_2\boldsymbol{c_2}, \qquad \alpha_1, \alpha_2 \in \mathbb{R}$$

Hence,

$$\begin{aligned}
[\boldsymbol{R} - \boldsymbol{I_3}]\left(\frac{[\boldsymbol{I_3} - \boldsymbol{R^T}]\boldsymbol{d}}{3 - \text{trace}([\boldsymbol{R}])}\right) &= \frac{-[\boldsymbol{2I_3} - (\boldsymbol{R} + \boldsymbol{R^T})](\delta\hat{\boldsymbol{k}} + \alpha_1\boldsymbol{c_1} + \alpha_2\boldsymbol{c_2})}{3 - (1 + 2\cos\phi)} \\
&= \frac{-(2 - (2\cos\phi))(\alpha_1\boldsymbol{c_1} + \alpha_2\boldsymbol{c_2})}{2(1-\cos\phi)} \\
&= -(\alpha_1\boldsymbol{x_1} + \alpha_2\boldsymbol{x_2}) = -\boldsymbol{d_p}
\end{aligned}$$

In the steps above we have used the fact that $i)$ the $\text{trace}([\boldsymbol{R}]) = 1 + 2\cos\phi$, $ii)$ $[\boldsymbol{R}]^{\boldsymbol{T}}\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}$, and that $iii)$ $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ behave identically as in eqs. (4) and (5), when multiplied by $[\boldsymbol{R}]^{\boldsymbol{T}}$, except that ϕ should be replaced by $-\phi$.

Secondly, in order to show that $\boldsymbol{r_0}$ is perpendicular to the screw axis, we note that the $\text{range}([\boldsymbol{I_3} - \boldsymbol{R^T}]) = \text{range}([\boldsymbol{I_3} - \boldsymbol{R}])$ is the plane passing through origin and perpendicular to $\boldsymbol{k}$. Clearly $\boldsymbol{r_0}$ is in the range space of $[\boldsymbol{I_3} - \boldsymbol{R^T}]$ and hence perpendicular to the screw axis which is along $\hat{\boldsymbol{k}}$.

From the above two arguments, $\boldsymbol{r_0}$ is a point on the screw axis and is closest to the origin.

The formula for $\boldsymbol{r_0}$ in eq. (15) is equivalent to the formula for "pole $\boldsymbol{c}$ of planar displacement", in terms of dot and cross product, given in eq. (1.65) on page 20 of McCarthy(1990). However, our formula is simpler since $\boldsymbol{r_0}$ can be derived directly from the elements of $[\boldsymbol{T}]$. In addition, we can clearly see the limiting case of $\phi \to 0$ from eq. (15). For $\phi \to 0$, we have

$$\lim_{\phi\to 0}\frac{[\boldsymbol{I_3} - \boldsymbol{R^T}]\boldsymbol{d}}{2(1-\cos\phi)} = \lim_{\phi\to 0}\left(\frac{\alpha_1\boldsymbol{c_1}}{2} + \frac{\alpha_2\boldsymbol{c_2}}{2} + \frac{2\alpha_1\boldsymbol{c_2}\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{4\sin^2\frac{\phi}{2}} - \frac{2\alpha_2\boldsymbol{c_1}\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{4\sin^2\frac{\phi}{2}}\right)$$

As long as either α_1 or α_2 is nonzero and finite, the above limit is ∞. This is to be expected, since $\phi \to 0$ implies that the motion is close to pure translation and the translation $\boldsymbol{d_p}$ can be accomplished by infinitesimally small rotations about an axis at infinity.

From the formula of $\boldsymbol{r_0}$, we are in a position to give the following simple choice for the basis of the Jordan form

$$\tilde{\boldsymbol{f}}_{\boldsymbol{1}} = (\boldsymbol{x_1}, 0),\ \tilde{\boldsymbol{f}}_{\boldsymbol{2}} = (\boldsymbol{x_2}, 0),\ \tilde{\boldsymbol{f}}_{\boldsymbol{3}} = (\hat{\boldsymbol{k}}, 0),\ \tilde{\boldsymbol{f}}_{\boldsymbol{4}} = \frac{(\boldsymbol{r_0}, 1)}{\delta}$$

When $\delta = 0$, we can choose the eigenbasis

$$\tilde{\boldsymbol{f_1}} = (\boldsymbol{x_1}, 0),\ \tilde{\boldsymbol{f_2}} = (\boldsymbol{x_2}, 0),\ \tilde{\boldsymbol{f_3}} = (\hat{\boldsymbol{k}}, 0),\ \tilde{\boldsymbol{f_4}} = (\boldsymbol{r_0}, 1)$$

which will also serve as the simple change of basis to obtain the form $[\boldsymbol{C}](\phi, \delta)$, shown in eq. (14).

Finally, if $[\boldsymbol{R}] = [\boldsymbol{I_3}]$, then we can have two cases: a) $\boldsymbol{d} = 0$ and b)$\boldsymbol{d} \neq 0$. For $\boldsymbol{d} = 0$, $[\boldsymbol{T}]$ is simply an identity matrix. If $\boldsymbol{d} \neq \boldsymbol{0}$, then the Jordan form is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last two basis for the Jordan form are

$$\tilde{\boldsymbol{f_3}} = \gamma(\boldsymbol{d}, 0),\ \tilde{\boldsymbol{f_4}} = \gamma(0, 0, 0, 1) + (\boldsymbol{q}, 0)$$

where $\boldsymbol{q}$ is any arbitrary vector in $\mathbb{R}^{\boldsymbol{3}}$ and γ is arbitrary but nonzero. The first two basis can be of the form $\tilde{\boldsymbol{f_1}} = (\boldsymbol{a}, 0)$ and $\tilde{\boldsymbol{f_2}} = (\boldsymbol{b}, 0)$, where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{d}$ are linearly independent in $\mathbb{R}^{\boldsymbol{3}}$.

3.1 Useful Implications of $[\boldsymbol{C}](\phi, \delta)$ and $[\boldsymbol{A}](\hat{\boldsymbol{k}}, \boldsymbol{r_0})$

Here we list some of the useful implications of the form $[\boldsymbol{C}](\phi, \delta)$ and its corresponding basis. To this end, we shall denote the change of basis matrix corresponding to form $[\boldsymbol{C}](\phi, \delta)$ as $[\boldsymbol{A}]$, i.e. $[\boldsymbol{A}] = [(\boldsymbol{x_1}, 0)\ (\boldsymbol{x_2}, 0)\ (\hat{\boldsymbol{k}}, 0)\ (\boldsymbol{r_0}, 1)]$.

1. For a given $[\boldsymbol{T}]$ we can obtain the the screw parameters, ($\hat{\boldsymbol{k}}$, $\boldsymbol{r_0}$, ϕ, δ), from any of the known methods. From the screw parameters one can obtain $[\boldsymbol{C}](\phi, \delta)$ (see eq. (14)) and $[\boldsymbol{A}]$ which satisfy the relationship $[\boldsymbol{C}](\phi, \delta) = [\boldsymbol{A}]^{-\boldsymbol{1}}[\boldsymbol{T}][\boldsymbol{A}]$. This relationship can be used to obtain the transformation matrix, given the screw parameters, as $[\boldsymbol{T}] = [\boldsymbol{A}][\boldsymbol{C}](\phi, \delta)[\boldsymbol{A}]^{-\boldsymbol{1}}$. Hence the modified Jordan form and its basis provides an alternate way of finding the transformation matrix, given the screw parameters. [5]

2. Consider a transformation matrix $[\boldsymbol{T}]$ with fixed $\hat{\boldsymbol{k}}$ and $\boldsymbol{r_0}$ and variable ϕ and δ. For such a transformation matrix, $[\boldsymbol{A}]$ is constant, and the change in the transformation matrix due to variation in ϕ and δ can be more conveniently effected by first obtaining the canonical form $[\boldsymbol{C}](\phi, \delta)$ (by the similarity transformation $[\boldsymbol{C}](\phi, \delta) = [\boldsymbol{A}]^{-\boldsymbol{1}}[\boldsymbol{T}]\boldsymbol{(\phi, \delta)}[\boldsymbol{A}]$), performing the required variation in ϕ and δ on $[\boldsymbol{C}](\phi, \delta)$ and finally converting back to $[\boldsymbol{T}](\phi, \delta)$ by the inverse similarity transformation $[\boldsymbol{T}](\phi, \delta) = [\boldsymbol{A}][\boldsymbol{C}]\boldsymbol{(\phi, \delta)}[\boldsymbol{A}]^{-\boldsymbol{1}}$.

   If δ is a linear function of ϕ, say $\delta = \beta\phi$, where β is the pitch, then $[\boldsymbol{C}](\phi, \beta\phi)$ has an exponential form $[\boldsymbol{C}](\phi, \beta\phi) = e^{[\boldsymbol{V}]\phi}$, where

$$[\boldsymbol{V}] = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[5]The quantities $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ that appear in $\tilde{\boldsymbol{f_1}}$ and $\tilde{\boldsymbol{f_2}}$ could always be found if $\boldsymbol{k}$ is known as illustrated in the example of Section 4.

This can be verified by expanding $e^{[\boldsymbol{V}]\phi}$ as $[\boldsymbol{I_4}] + \frac{[\boldsymbol{V}]\phi}{1\,!} + \frac{[\boldsymbol{V}]^{\boldsymbol{2}}\phi^2}{2\,!} + ...$ and using the fact that

$$[\boldsymbol{V}]^{\boldsymbol{n}} = \begin{bmatrix} i^n & 0 & 0 & 0 \\ 0 & -i^n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for} \quad n > 1.$$

From the exponential form for $[\boldsymbol{C}](\phi, \beta\phi)$, the transformation matrix $[\boldsymbol{T}]$ can also be expressed in an exponential form as

$$[\boldsymbol{T}](\phi, \beta\phi) = [\boldsymbol{A}][\boldsymbol{C}](\phi, \beta\phi)[\boldsymbol{A}]^{-\boldsymbol{1}} = [\boldsymbol{A}]e^{[\boldsymbol{V}]\phi}[\boldsymbol{A}]^{-\boldsymbol{1}} = e^{([\boldsymbol{A}][\boldsymbol{V}][\boldsymbol{A}]^{-\boldsymbol{1}})\phi}$$

where the last equality can be verified by expanding the exponential function.

The result $[\boldsymbol{T}](\phi, \beta\phi) = e^{([\boldsymbol{A}][\boldsymbol{V}][\boldsymbol{A}]^{-\boldsymbol{1}})\phi}$ is strikingly similar to an exponential form of a rotation matrix given as $[\boldsymbol{R}](\phi) = e^{[\widetilde{\boldsymbol{k}}]\phi}$ (see, for example, Problem 2.15, pp. 62 in Craig 1989).
The quantity $[\boldsymbol{A}][\boldsymbol{V}][\boldsymbol{A}]^{-\boldsymbol{1}}$ could be further simplified by writing $[\boldsymbol{A}]$ in the block matrix form

$$[\boldsymbol{A}] = \begin{bmatrix} [\boldsymbol{G}] & \boldsymbol{r_0} \\ [0\ 0\ 0] & 1 \end{bmatrix}$$

Without loss of generality, we assume $[\boldsymbol{G}] = [\hat{\boldsymbol{x}}_{\boldsymbol{1}}\ \hat{\boldsymbol{x}}_{\boldsymbol{2}}\ \hat{\boldsymbol{k}}]$ where $\hat{\boldsymbol{x}}_{\boldsymbol{1}} = \boldsymbol{c_1} - \imath\boldsymbol{c_2}$ and $\hat{\boldsymbol{x}}_{\boldsymbol{1}} = \boldsymbol{c_1} + \imath\boldsymbol{c_2}$, with $|\boldsymbol{c_1}|^2 = |\boldsymbol{c_2}|^2 = 1/2$ so that $[\boldsymbol{G}]$ becomes a unitary matrix. Under this condition

$$[\boldsymbol{A}]^{-\boldsymbol{1}} = \begin{bmatrix} [\overline{\boldsymbol{G}}]^{\boldsymbol{T}} & -[\overline{\boldsymbol{G}}]^{\boldsymbol{T}}\boldsymbol{r_0} \\ [0\ 0\ 0] & 1 \end{bmatrix}$$

where $[\overline{\boldsymbol{G}}]^{\boldsymbol{T}}$ is the conjugate transpose or hermitian transpose of $[\boldsymbol{G}]$. We can also write $[\boldsymbol{V}]$ in the form

$$[\boldsymbol{V}] = \begin{bmatrix} [\boldsymbol{F}] & \boldsymbol{b} \\ [0\ 0\ 0] & 0 \end{bmatrix}, \quad [\boldsymbol{F}] = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix}$$

Therefore, we have

$$[\boldsymbol{A}][\boldsymbol{V}][\boldsymbol{A}]^{-\boldsymbol{1}} = \begin{bmatrix} [\boldsymbol{G}][\boldsymbol{F}][\overline{\boldsymbol{G}}]^{\boldsymbol{T}} & -[\boldsymbol{G}][\boldsymbol{F}][\overline{\boldsymbol{G}}]^{\boldsymbol{T}}\boldsymbol{r_0} + \beta\hat{\boldsymbol{k}} \\ [0\ 0\ 0] & 0 \end{bmatrix}$$

By performing the multiplication $[\boldsymbol{G}][\boldsymbol{F}][\overline{\boldsymbol{G}}]^{\boldsymbol{T}}$ and using $\hat{\boldsymbol{k}} = 2(\boldsymbol{c_1} \times \boldsymbol{c_2})$, we get $[\boldsymbol{G}][\boldsymbol{F}][\overline{\boldsymbol{G}}]^{\boldsymbol{T}} = [\widetilde{\boldsymbol{k}}]$. Hence the simplified form of $[\boldsymbol{A}][\boldsymbol{V}][\boldsymbol{A}]^{-\boldsymbol{1}}$ is given by

$$[\boldsymbol{A}][\boldsymbol{V}][\boldsymbol{A}]^{-\boldsymbol{1}} = \begin{bmatrix} [\widetilde{\boldsymbol{k}}] & -[\widetilde{\boldsymbol{k}}]\boldsymbol{r_0} + \beta\hat{\boldsymbol{k}} \\ [0\ 0\ 0] & 1 \end{bmatrix}$$

4 Numerical Example

In many cases it is required to move a rigid body from one configuration to another by pure rotation about an axis. A typical example is that of an antenna or a deployable structure in a satellite which are stowed during launch and later deployed and locked in orbit. This is generally accomplished using multiple tilts with respect to satellite (Nagaraj *et* al. 1996). A single screw joint can be used as long as there is no pure translation. However, it would be nice to know, if the motion could be accomplished by a single motorised or spring energized revolute joint, since a single revolute joint would be simpler with the advantage of lesser friction over a screw joint. Based on the concepts introduced in the earlier sections, one can easily check whether a single revolute joint is feasible, and if feasible, obtain the location where the revolute joint must be placed. If a single revolute joint is not feasible, we can again locate the axis of the screw joint. We illustrate the two cases by a numerical example taken from (Nagaraj *et* al. 1996) dealing with obtaining the hinge line for deployment of an antenna from a stowed position in orbit. In the numerical example we also show how the basis shown at the end of Section 3 reduces the 4×4 transformation matrix to Jordan or diagonal form for the two cases. All the computations were performed in Matlab.

Consider the following transformation matrix

$$[\boldsymbol{T}] = \begin{bmatrix} 0.4490 & 0.5463 & -0.7071 & 0.0239 \\ 0.4490 & 0.5463 & 0.7071 & 0.0739 \\ 0.7725 & -0.6350 & -0.0000 & -0.1214 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix}$$

The rank of $[\boldsymbol{T} - \boldsymbol{I_4}]$ turn out to be 3. By rank-nullity theorem, nullity$([\boldsymbol{T} - \boldsymbol{I_4}]) = 1$, and hence we conclude that the rigid displacement, under consideration cannot be accomplished by a single revolute joint. However, we can find the location of the screw axis from eq. (15). Other quantities related to $[\boldsymbol{T}]$ have also been found and tabulated below.

	Identity used	Numerical Values
$\boldsymbol{r_0}$	$\boldsymbol{r_0} = \frac{[\boldsymbol{I_3} - \boldsymbol{R^T}]\boldsymbol{d}}{3 - \text{trace}([\boldsymbol{R}])}$	(0.0368, -0.0282, -0.0782)
ϕ	$\text{trace}([\boldsymbol{R}]) = 1 + 2\cos\phi$	1.5732 radian
$\hat{\boldsymbol{k}}$	vector form of the skew symmetric matrix $\frac{([\boldsymbol{R} - \boldsymbol{R^T}])}{(2\sin\phi)}$	(-0.6711, -0.7398, -0.0486)
δ	$\delta = \hat{\boldsymbol{k}} \cdot \boldsymbol{d}$	-0.0648 unit of length

The vector $\boldsymbol{c_1}$ is found using $\boldsymbol{c_1} = (\boldsymbol{a} - (\hat{\boldsymbol{k}} \cdot \boldsymbol{a})\hat{\boldsymbol{k}})/|\boldsymbol{a} - (\hat{\boldsymbol{k}} \cdot \boldsymbol{a})\hat{\boldsymbol{k}}|$ with $\boldsymbol{a}$ being linearly independent from $\hat{\boldsymbol{k}}$ but otherwise arbitrary. For $\boldsymbol{a} = (1, 0, 0)$, we get $\boldsymbol{c_1} = (0.7414, -0.6696, -0.0440)$. The vector $\boldsymbol{c_2}$ is found using $\boldsymbol{c_2} = \hat{\boldsymbol{k}} \times \boldsymbol{c_1}$ which gives $\boldsymbol{c_2} = (0, -0.0656, 0.9978)$. From $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$, we obtain $\boldsymbol{x_1} = \boldsymbol{c_1} - i\boldsymbol{c_2}$ and $\boldsymbol{x_2} = \boldsymbol{c_1} + i\boldsymbol{c_2}$ (for the relationship between $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ see page 14 in McCarthy (1990)). Using the basis for the Jordan form given in Section 3, we obtain the following change of basis matrix

$$[\boldsymbol{B}] = \begin{bmatrix} 0.7414 & 0.7414 & -0.6711 & -0.5678 \\ -0.6696 + 0.0656i & -0.6696 - 0.0656i & -0.7398 & 0.4357 \\ -0.0440 - 0.9978i & -0.0440 + 0.9978i & -0.0486 & 1.2065 \\ 0 & 0 & 0 & -15.4302 \end{bmatrix}$$

and the following Jordan form (from the similarity transformation)

$$[\boldsymbol{B}]^{-1}[\boldsymbol{T}][\boldsymbol{B}] = \begin{bmatrix} -0.0024 + 1.0000i & 0 & 0 & 0 \\ 0 & -0.0024 - 1.0000i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The upper-left 2×2 sub-matrix has $e^{\imath\phi}$ and $e^{-\imath\phi}$ as diagonal elements. The ϕ obtained from the sub-matrix is 1.5732. This value agrees with the value of ϕ found earlier in the above table.

Now suppose we are allowed to add to the existing $\boldsymbol{d}$, a scalar multiple of a vector, say (1,1,1), such that δ becomes zero. In such a case the scalar multiple α can be obtained from $\alpha = -\delta/(\hat{\boldsymbol{k}} \cdot (1,1,1))$, and α evaluates to -0.0444. The modified transformation matrix with the rotation part remaining unchanged is given by

$$[\boldsymbol{T_d}] = \begin{bmatrix} 0.4490 & 0.5463 & -0.7071 & 0.0205 \\ 0.4490 & 0.5463 & 0.7071 & 0.0295 \\ 0.7725 & -0.6350 & -0.0000 & -0.1658 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix}$$

The rank of $[\boldsymbol{T_d} - \boldsymbol{I_4}]$ turns out to be 2 and by rank nullity theorem, its nullity is 2. Hence the rigid body displacement represented by $[\boldsymbol{T_d}]$ can be accomplished by a single revolute joint. For this modified $[\boldsymbol{T}]$, $\boldsymbol{r_0}$ is given by $(0.0517, -0.0403, -0.1003)$ which together with $\hat{\boldsymbol{k}}$ locates the 'hinge-line'.

We can obtain the change of basis matrix using the basis suggested at end of Section 3 and perform the similarity transformation to obtain the diagonal form. This is shown below.

$$[\boldsymbol{A}] = \begin{bmatrix} 0.7414 & 0.7414 & -0.6711 & 0.0517 \\ -0.6696 + 0.0656i & -0.6696 - 0.0656i & -0.7398 & -0.0403 \\ -0.0440 - 0.9978i & -0.0440 + 0.9978i & -0.0486 & -0.1003 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix}$$

$$[\boldsymbol{A}]^{-1}[\boldsymbol{T_d}][\boldsymbol{A}] = \begin{bmatrix} -0.0024 + 1.0000i & 0 & 0 & 0 \\ 0 & -0.0024 - 1.0000i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5 Conclusion

In this paper, we have shown that the 4×4 homogeneous transformation matrix can be diagonalized if and only if the rigid body displacement associated with the transformation matrix is degenerate with the displacement along the screw axis equal to zero. We have also presented explicit expressions for the Jordan basis when the transformation matrix is not diagonalizable, and presented a novel formula for the closest point on the screw axis from the origin. The theoretical results are illustrated by a numerical example. The 4×4 homogeneous transformation matrix is one of the most widely used matrix in the area of kinematics, robotics and CAD, in this note we have a presented a rigorous algebraic treatment and some non-trivial properties of these matrices.

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