Analytical determination of principal twists in serial, parallel and hybrid manipulators using dual vectors and matrices

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Abstract

The determination of principal twists of the end-effector of a multi-degree-of-freedom manipulator play a central role in their analysis, design, motion planning and determination of singularities. Most approaches to obtain principal twists and the distributions of twists, such as the well-known classical results of cylindroid and hyperboloid, are based on geometric reasoning and involve intuitive choice of coordinate systems. In this paper, we present a formal algebraic approach to obtain the principal twists of any multi-degree-of-freedom serial, parallel or hybrid manipulator, by making use of the algebra of dual numbers, vectors and matrices. We present analytical expressions for the principal twists and the pitches for any arbitrary degree-of-freedom manipulator. A consequence of our approach is that we can obtain analytical expressions for the screws along which a manipulator can lose or gain degrees-of-freedom at a singularity. The theoretical results are illustrated with the help of examples of parallel and hybrid manipulators.

1 Introduction

The analysis of rigid-body displacement and motion may be done formally within the framework of group theory and Lie algebra. It is well known in literature that rigid-body displacements constitute a group of isometries of $\Re^3$, known as the *Special Euclidean Group* (denoted by $SE(3)$). $SE(3)$ is also a 6-dimensional smooth manifold, and therefore it is a Lie group. The quantities of our interest, i.e., the rigid-body twists, lie in tangent-space at the identity of $SE(3)$, which is also the lie algebra (denoted by $se(3)$) associated with this group (see, e.g., [1]). Twists are 6-dimensional, but using the Plücker vectors, they may be expressed conveniently in terms of a pair of vectors in $\Re^3$. The central problem of screw theory is to find a suitable *principal* basis of $\Re^3$, in which the Plücker vectors representing the *basis vectors* of $se(3)$, or the *principal screws* take the simplest of forms, and provide maximum amount of information regarding the distribution of screws explicitly. However, it is obvious that such a basis of $\Re^3$ would be *non-unique*, since the dimension of $se(3)$ is 6, and there are free choices regarding the definition of the *principal basis*.

Most of the existing literature restricts the magnitude of the twists to unity, and analyze the resulting space of screws, $\mathcal{P}^5$, rather than $se(3)$ in its completeness. The principal screws are obtained using geometric arguments[2, 3]. We, however, make use of the group structure of $SE(3)$, and using the dual orthogonal matrix representation of $SE(3)$[4], arrive at the dual vector representation of its Lie algebra elements. Dual 3-vectors form a free-module, $\mathcal{D}^3$, over the ring of dual numbers (denoted by Δ)[5], which is isomorphic to $se(3)$. This allows us to define an *inner product* on $se(3)$, and the corresponding *norm* of twists, as an element of Δ. Using these concepts, we reduce the problem of identification of principal twists to the extremization of the norm of the resultant twist.

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The extremization leads to an *eigen problem* of a symmetric dual matrix, consisting of the dual inner-products of the input screws, which is positive semi-definite over Δ.

We show that our formulation results in analytical description of two different *principal bases*, which contains all the relevant information about the first-order kinematics. Extremization of the real part of the dual norm leads to a basis (denoted by $\boldsymbol{\omega}-$basis), in which the degrees-of-freedom of a multi-degrees-of-freedom rigid-body motion decouples into pure translational and finite-pitch screw modes. We also show that the classical basis of Ball, which we denote as the $\boldsymbol{h}-$basis, and the $\boldsymbol{\omega}-$basisarise out of the general case of the dual eigen problem. We show that the determination of the principal screws of the $\boldsymbol{h}-$basis reduces to a generalized eigen problem involving the real and dual parts of a dual matrix. The principal pitches are proved to be half of the generalized eigenvalues, and analytical expressions for them can be obtained in closed form. These results form the most important contributions of the paper, and the theoretical development and the analytical results are more complete and elegant than the previous attempts of algebraic formulation of principal screws (see, e.g,[6, 7]).

The significant advantages of our approach are:

1. The approach is *exact*, i.e., all the results are obtained in closed, analytical form. This is possible since the dual eigenvalue problem leads to a dual characteristic polynomial which is at most a cubic and this can be solved analytically.

2. Our treatment of rigid-body motion leads to the identification of the rotational and translational degrees-of-freedom separately, which we term as *DOF-partitioning*. The concept of DOF-partitioning, in conjunction with our analytical approach, is useful for analysis and design of manipulators. In particular, singularities leading to loss and gain of one or more degrees-of-freedom in serial, parallel and hybrid manipulators can be handled in a natural way, and analytical identification of the lost or gained twists at a singularity is possible.

3. The computations involved are of purely algebraic nature, and hence amenable to fast and automated computer implementation. All symbolic manipulations are done using `Mathematica`[8].

The paper is organized as follows: in section 2, we present, briefly, the representation of lines, screws and twists using dual numbers and vectors, introduce the notion of an inner product of two dual vectors as a dual number, and a matrix of dual inner products. We also present salient properties of the dual matrix and show how the identification of principal twists leads to a dual eigenvalue problem. We show how the eigenvalue problem leads naturally to first the $\boldsymbol{\omega}-$basisand then the $\boldsymbol{h}-$basis. In section 3, we present the analytical expressions of the principal twists for multi-degree-of-freedom rigid body motions and the concept of partitioning of degrees-of-freedom of a rigid body. In section 4, we present a discussion on the analytical identification of lost or gained twists at a singularity. The theoretical results developed in section 3 and 4, are illustrated in section 5 with the help of a parallel and a hybrid manipulator. In the appendices, we present derivation of equivalent Jacobian for parallel and hybrid manipulators and the details of the solution of the dual eigenvalue problem.

2 Mathematical formulation

The theoretical development and results presented in this paper are obtained using dual numbers, vectors and matrices. In this section, we first briefly review the notions of a dual number, vector and a matrix, and then apply them to analyze multi-degree-of-freedom rigid body motion.

2.1 Lines, screws, and twists as dual vectors

A dual number, $\hat{a}$, has the form $a + \epsilon a_0$, where $a,\ a_0 \in \Re$ and ϵ stands for the *dual unit*, with the properties $\epsilon \neq 0,\ \epsilon^2 = 0$. The properties of dual numbers are detailed in[9]. We note here only that the dual numbers over the real field form a *ring*, and dual n-vectors form a *free module* over this ring[5], which is denoted by $\mathcal{D}^n$. We can define an inner product on $\mathcal{D}^3$, the space of 3-dimensional dual vectors, as follows:

$$\langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle = \boldsymbol{x} \cdot \boldsymbol{y} + \epsilon(\boldsymbol{x} \cdot \boldsymbol{y}_0 + \boldsymbol{y} \cdot \boldsymbol{x}_0) = -\frac{1}{4} \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle_{Killing} + \epsilon \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle_{Klein} \tag{1}$$

where '$\cdot$' denotes the usual inner product in the Euclidean space, $\hat{\boldsymbol{x}} = \boldsymbol{x} + \epsilon \boldsymbol{x}_0 \ \in \mathcal{D}^3$ and $\langle \cdot, \cdot \rangle_{Killing}$ and $\langle \cdot, \cdot \rangle_{Klein}$ are the *Killing* and *Klein forms* on $SE(3)$ respectively[1]. Both these forms are known to posses frame-invariance, and hence the dual inner product is frame-invariant. The inner product is positive semi-definite, as the Killing form is negative semi-definite. Using the inner-product, we can define the norm $\|\hat{\boldsymbol{x}}\|$ of $\hat{\boldsymbol{x}}$ as $\langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{x}} \rangle^{1/2}$ when $\boldsymbol{x} \neq \boldsymbol{0}$. Then we obtain $\|\hat{\boldsymbol{x}}\| = \|\boldsymbol{x}\| + \epsilon \frac{\boldsymbol{x} \cdot \boldsymbol{x}_0}{\|\boldsymbol{x}\|},\quad \|\boldsymbol{x}\| \neq \boldsymbol{0}$. A dual vector $\hat{\boldsymbol{x}}$ with norm $1 + \epsilon 0$ is called a *dual unit vector*[9], and follows the relations $\|\boldsymbol{x}\| = 1, \boldsymbol{x}.\boldsymbol{x}_0 = 0$. A line in $\Re^3$ can be described in terms of a dual unit vector as $\hat{\mathcal{L}} = \boldsymbol{Q} + \epsilon \boldsymbol{Q}_0$, where $(\boldsymbol{Q}; \boldsymbol{Q_0})$ is the Plücker vector associated with the line (see, e.g., [10]). There is a one-to-one correspondence between lines in $\Re^3$ and Plücker vectors, since $\boldsymbol{Q}$ is a unit vector giving the direction of the line, and its location is uniquely determined by the foot of the perpendicular from the origin, $\boldsymbol{r}_0 = \boldsymbol{Q} \times \boldsymbol{Q}_0$. The inner product of two lines follows from the properties of dual vectors, and is given by

$$\left\langle \hat{\mathcal{L}}_1, \hat{\mathcal{L}}_2 \right\rangle = \boldsymbol{Q}_1 \cdot \boldsymbol{Q}_2 + \epsilon(\boldsymbol{Q}_1 \cdot \boldsymbol{Q}_{02} + \boldsymbol{Q}_2 \cdot \boldsymbol{Q}_{01}) = \cos\phi - \epsilon d \sin\phi = \cos\hat{\phi} \tag{2}$$

where ϕ and d are the angle and the shortest distance between the two lines respectively, and $\hat{\phi} = \phi + \epsilon d$ denotes the dual angle between the lines [11].

A *screw* has five independent parameters and can be described by a dual vector $\hat{\$} = \boldsymbol{S} + \epsilon \boldsymbol{S}_0$, where $\boldsymbol{S} = \boldsymbol{Q}$ and $\boldsymbol{S}_0 = \boldsymbol{Q}_0 + h\boldsymbol{Q}$. The pitch of the screw, h, is given by $h = \frac{\boldsymbol{S} \cdot \boldsymbol{S}_0}{\boldsymbol{S} \cdot \boldsymbol{S}},\quad \|\boldsymbol{S}\| \neq 0$. If the magnitude of the real part of $\hat{\$}$ is 0, and that of the dual part is non-zero, then the pitch is infinite, signifying a pure translation. The inner product of two screws is computed as

$$\left\langle \hat{\$}_1, \hat{\$}_2 \right\rangle = \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + \epsilon(\boldsymbol{S}_1 \cdot \boldsymbol{S}_{02} + \boldsymbol{S}_2 \cdot \boldsymbol{S}_{01}) = \cos\phi + \epsilon((h_1 + h_2)\cos\phi - d\sin\phi) \tag{3}$$

where h_1 and h_2 are the pitches associated with the two screws respectively.

We parameterize $SE(3)$, the space of rigid-body displacements, in terms of dual orthogonal matrices of the form $\hat{\boldsymbol{A}} = \boldsymbol{R} + \epsilon \boldsymbol{D}\boldsymbol{R}$, where $\boldsymbol{R} \in SO(3)$ gives the orientation of the moving frame attached to the rigid-body with respect to some fixed reference frame, and $\boldsymbol{D} \in so(3)$ is the 3×3 skew-symmetric matrix associated with $\boldsymbol{d} \in \Re^3$, the displacement of the origin of the moving frame with respect to the fixed frame[4]. For n-DOF motions of the rigid-body, we can associate n independent real *motion parameters*, $\theta_i, i = 1, \ldots n$ via a smooth map, $\boldsymbol{\psi} : \Re^n \to SE(3)$ such that $\boldsymbol{\psi}(\boldsymbol{\theta}) = \hat{\boldsymbol{A}} \in SE(3)$. The motion parameters, $\boldsymbol{\theta}$, may be assumed to be functions of time t alone, and thus the vector function $\boldsymbol{\theta}(t)$ describes the motion in $\Re^n$. As $\boldsymbol{\theta}(t)$ evolves smoothly, it traces a *curve* $c(t) = \boldsymbol{\psi}(\boldsymbol{\theta}(t))$ on the manifold $SE(3)$, to each point of which we can associate a tangent space containing the velocity $\dot{c}(t)$ of the curve. The tangent vector $\dot{c}(t)$ may be obtained from the push-forward map $\boldsymbol{\psi}_* : \Re^n \to T_{\hat{\boldsymbol{A}}}SE(3)$ such that $\boldsymbol{\psi}_*(\dot{\boldsymbol{\theta}}) = \dot{\hat{\boldsymbol{A}}}(\boldsymbol{\theta}(t)) = \dot{\boldsymbol{R}} + \epsilon(\dot{\boldsymbol{D}}\boldsymbol{R} + \boldsymbol{D}\dot{\boldsymbol{R}}) \in T_{\hat{\boldsymbol{A}}}SE(3)$. We can translate this tangent vector to the tangent-space at the *identity* element of $SE(3)$ by left or right translations by $\hat{\boldsymbol{A}}^{-1}(=\hat{\boldsymbol{A}}^T)$ to obtain the Lie algebra $se(3)$ associated with the group, where the multiplication is given by the Lie bracket, denoted by $[\cdot,\cdot]$. Elements of $se(3)$ are isomorphic to twists[1]. Depending upon the translation used to take them to the identity, we can get a *left-invariant* twist or a *right-invariant* twist. In this paper, we use the right-invariant twists[1], whose explicit form is $\hat{\boldsymbol{\Omega}} = \dot{\hat{\boldsymbol{A}}}\hat{\boldsymbol{A}}^T = \boldsymbol{\Omega} + \epsilon([\boldsymbol{D},\boldsymbol{\Omega}] + \dot{\boldsymbol{D}})$ where $\boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^T \in so(3)$ denotes the right-invariant angular velocity of the rigid-body. Using the isomorphism of the algebras $(so(3),[\cdot,\cdot])$ and $(\Re^3,\times)$, we express the twist in terms of a dual vector as

$$\hat{\mathcal{V}} = \boldsymbol{\omega} + \epsilon(\dot{\boldsymbol{d}} + \boldsymbol{d}\times\boldsymbol{\omega}) \tag{4}$$

where $\boldsymbol{\omega}, \dot{\boldsymbol{d}}$ and $\boldsymbol{d}\times\boldsymbol{\omega}$ are the counterparts of $\boldsymbol{\Omega}, \dot{\boldsymbol{D}}$, and $[\boldsymbol{D},\boldsymbol{\Omega}]$ respectively in $\Re^3$. The quantity $\hat{\boldsymbol{\mathcal{V}}}$ is also known as a *motor*, and may be thought of as a screw together with a magnitude[11]. In terms of line coordinates, $\hat{\boldsymbol{\mathcal{V}}} = ||\boldsymbol{\omega}||(\boldsymbol{Q} + \epsilon(\boldsymbol{Q}_0 + h\boldsymbol{Q}))$, where $||\boldsymbol{\omega}||$, the magnitude of the angular velocity vector, also denotes the magnitude of the twist.

The resultant twist of the end-effector of a n-DOF manipulator can be expressed as a linear combination of the input screws, $\hat{\$}_i, i = 1,...,n$, as

$$\hat{\boldsymbol{\mathcal{V}}} = \sum_{i=1}^{n} \hat{\$}_i\dot{\theta}_i = \hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}} = \boldsymbol{J}_{\boldsymbol{\omega}}\dot{\boldsymbol{\theta}} + \epsilon\boldsymbol{J}_{\boldsymbol{v}}\dot{\boldsymbol{\theta}},\ i = 1,\ldots,n \tag{5}$$

where $\hat{\$}_i$, the ith column of $\hat{\boldsymbol{J}}$, may be computed as the vector form of the dual skew-symmetric matrix $\frac{\partial\hat{\boldsymbol{A}}}{\partial\theta_i}\hat{\boldsymbol{A}}^T$, and θ_i is the joint variable corresponding to the ith joint. The dual Jacobian $\hat{\boldsymbol{J}}$ is composed of the Jacobians $\boldsymbol{J}_{\boldsymbol{\omega}}$ and $\boldsymbol{J}_{\boldsymbol{v}}$ corresponding to the angular and linear velocities respectively[2]. The square of the dual *norm* of $\hat{\boldsymbol{\mathcal{V}}}$ may be written as

$$||\hat{\boldsymbol{\mathcal{V}}}||^2 = \dot{\boldsymbol{\theta}}^T\hat{\boldsymbol{g}}\dot{\boldsymbol{\theta}} = ||\boldsymbol{\omega}||^2(1+\epsilon(2h)) \tag{6}$$

where $||\boldsymbol{\omega}||$ is the magnitude of the twist, and h is its pitch. The matrix $\hat{\boldsymbol{g}}$ is defined by the equation

$$\hat{\boldsymbol{g}} = \boldsymbol{g} + \epsilon\boldsymbol{g}_0 = \hat{\boldsymbol{J}}^T\hat{\boldsymbol{J}} = \boldsymbol{J}_{\boldsymbol{\omega}}^T\boldsymbol{J}_{\boldsymbol{\omega}} + \epsilon(\boldsymbol{J}_{\boldsymbol{\omega}}^T\boldsymbol{J}_{\boldsymbol{v}} + \boldsymbol{J}_{\boldsymbol{v}}^T\boldsymbol{J}_{\boldsymbol{\omega}}) \tag{7}$$

It may be noted that the (i,j)th element of $\hat{\boldsymbol{g}}$ is $\left\langle \hat{\$}_i, \hat{\$}_j \right\rangle$, the dual inner product of the ith and jth input screws. This implies that the matrix $\hat{\boldsymbol{g}}$ is *symmetric* and *frame invariant*. The frame-invariance follows from the fact that both the Klein and Killing forms are known to be frame invariant.

Following the results for point trajectories[12], we seek the extremal values of the square of the magnitudes of the resultant twist, $||\hat{\boldsymbol{\mathcal{V}}}||^2$, subject to a *unit speed* constraint, $||\dot{\boldsymbol{\theta}}|| = 1$. Using equation (6) and *Lagrange multipliers*, $\hat{\lambda}_i \in \Delta$, the objective function to be minimized is $\hat{g}_{ij}\dot{\theta}_i\dot{\theta}_j - \hat{\lambda}_i(\dot{\theta}_i^2 - 1)$, $\ i,j = 1,\ldots,n$. The solution of this n-dimensional extremization problem leads to the eigen problem

$$\hat{\boldsymbol{g}}\dot{\boldsymbol{\theta}} = \hat{\lambda}\dot{\boldsymbol{\theta}} \tag{8}$$

where $\hat{\lambda} = \lambda + \epsilon\lambda_0$ is the dual eigenvalue of $\hat{\boldsymbol{g}}$. Appendix B presents a procedure to solve the eigen problem associated with an arbitrary dual square matrix. In the following, we present the special properties of the eigen system of $\hat{\boldsymbol{g}}$ which forms the basis of the theoretical results of this paper.

[1]Analogous results can be obtained for left-invariant twists.

[2]The Jacobians of parallel and hybrid manipulators can be obtained as shown in Appendix A.

2.2 Properties of the eigen system of $\hat{g}$

Pre-multiplying equation (8) with $\dot{\boldsymbol{\theta}}^T$ and comparing with equation (6), we find that $\lambda = ||\boldsymbol{\omega}||^2$, and $\lambda_0 = 2||\boldsymbol{\omega}||^2 h$. When $||\boldsymbol{\omega}||^2 \neq 0$, the principal pitch can be obtained as

$$h = \frac{\lambda_0}{2\lambda} \tag{9}$$

The ring of dual numbers have the lexicographical order

$$\hat{x}_1 >< \hat{x}_2 \text{ if } \mathrm{x}_1 >< \mathrm{x}_2, \quad \text{if } \mathrm{x}_1 = \mathrm{x}_2, \text{then } \hat{\mathrm{x}}_1 >< \hat{\mathrm{x}}_2 \text{ if } \mathrm{x}_{01} >< \mathrm{x}_{02}$$

Therefore, the extremization of the magnitude of the resultant twist implies extremization of the real part of $||\hat{\boldsymbol{\mathcal{V}}}||^2$, i.e., $||\boldsymbol{\omega}||^2$. Expanding equation (8) into its real and dual parts, we get,

$$\begin{aligned} \boldsymbol{g}\dot{\boldsymbol{\theta}} &= \lambda\dot{\boldsymbol{\theta}} \\ \boldsymbol{g}_0\dot{\boldsymbol{\theta}} &= \lambda_0\dot{\boldsymbol{\theta}} \end{aligned} \tag{10}$$

Noting that $\boldsymbol{\omega} = \boldsymbol{J}_{\boldsymbol{\omega}}\dot{\boldsymbol{\theta}}$, it is easy to see from equations (6,7) that the extremization of $||\boldsymbol{\omega}||^2$ reduces to the first of equations (10) under the constraint $||\dot{\boldsymbol{\theta}}|| = 1$. However, if we consider the space of screws alone, with $\boldsymbol{\omega} = 1$, then as per the lexicographical order in Δ, $||\hat{\boldsymbol{\mathcal{V}}}||^2$ is extremized when its dual part or $2h$ is extremized, i.e., the pitches are extremal. In the following discussion, we elaborate on these two extremization, and show that they lead to two bases denoted by $\boldsymbol{\omega}-$basis and $\boldsymbol{h}-$basis.

2.3 Formulation of the $\boldsymbol{\omega}-$basis

We start with the first of equation (10), namely $\boldsymbol{g}\dot{\boldsymbol{\theta}} = \lambda\dot{\boldsymbol{\theta}}$. Since $rank_{\Re}\boldsymbol{J}_{\boldsymbol{\omega}} \leq 3$, the characteristic polynomial of $\boldsymbol{g}$, for arbitrary n DOF motion ($n \geq 3$), reduces to the form

$$\lambda^{n-3}(\lambda^3 - n\lambda^2 + a_{n-2}\lambda + a_{n-3}) = 0 \tag{11}$$

It maybe noted that we have to solve for at most a cubic, and the cubic is guaranteed to have real roots, since $\boldsymbol{g}$ is symmetric. It may be also noted that the cubic can be solved *analytically* in closed form using Cardan's formula[13]. Hence, we can obtain analytical expressions for the eigenvalues and the eigenvectors in terms of the input screw parameters. This key observation forms the basis of our theoretical results in this paper, and we obtain analytical expressions for the eigenvalues and eigenvectors for various n in the next section. The eigenvectors, denoted by $\dot{\boldsymbol{\theta}}_i^{\omega}$, form a basis of the row-space of $\hat{\boldsymbol{J}}$, and the principal twists, denoted by $\hat{\boldsymbol{\mathcal{V}}}_i^{\omega}$ lying in the column-space of $\hat{\boldsymbol{J}}$ may be obtained as

$$\hat{\boldsymbol{\mathcal{V}}}_i^{\omega} = \hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}}_i^{\omega} \tag{12}$$

The set $\{\hat{\boldsymbol{\mathcal{V}}}_i^{\omega}\}, \quad i = 1, \cdots, n$ constitute the $\boldsymbol{\omega}-$basis.

2.4 Formulation of the $\boldsymbol{h}-$basis

It is known from linear algebra, that equation (10) is consistent, i.e., the matrices $\boldsymbol{g}$ and $\boldsymbol{g}_0$ share an eigenvector, $\dot{\boldsymbol{\theta}}^h$, iff $\boldsymbol{g}\boldsymbol{g}_0 = \boldsymbol{g}_0\boldsymbol{g}$. In general, this condition will not be satisfied automatically. However, if $\boldsymbol{g}$ is positive definite we can always find a transformation T of $\Re^n$, n being the DOF of motion considered, which will reduce $\boldsymbol{g}$ and $\boldsymbol{g}_0$ to such forms that they commute. We consider here the case where $\boldsymbol{g}$ is positive definite and this happens when the DOF is 1, 2 or 3. In this case, the required transformation may be obtained in three steps:

1. Diagonalization of $\boldsymbol{g}$ and transformation of $\boldsymbol{g}_0$: The transformation T_1 has the eigenvectors of $\boldsymbol{g}$ as its columns and we can write

$$\begin{aligned} \boldsymbol{g}_1 &= T_1^{-1}\boldsymbol{g}T_1 = \mathrm{diag}\{\lambda_i\},\ \mathrm{i} = 1, \ldots, \mathrm{n} \\ \boldsymbol{g}_{01} &= T_1^{-1}\boldsymbol{g}_0 T_1 \end{aligned} \tag{13}$$

2. Scaling by the square-root of λ_i: The transformation T_2 is given by diag $\{1/\sqrt{\lambda_i}\}$, $i = 1, \ldots, n$, and we get

$$\begin{aligned} \boldsymbol{g}_2 &= T_2^{-1}\boldsymbol{g}_1 T_2 = \mathrm{diag}\{1, \ldots, 1\} \\ \boldsymbol{g}_{02} &= T_2^{-1}\boldsymbol{g}_{01} T_2 \end{aligned} \tag{14}$$

3. Diagonalization of $\boldsymbol{g}_{02}$ and transformation of $\boldsymbol{g}_2$: The transformation T_3 has the eigenvectors of $\boldsymbol{g}_{02}$ as its columns and we get

$$\begin{aligned} \boldsymbol{g}_3 &= T_3^{-1}\boldsymbol{g}_2 T_3 = \mathrm{diag}\{1, \ldots, 1\} \\ \boldsymbol{g}_{03} &= T_3^{-1}\boldsymbol{g}_{02} T_3 = \mathrm{diag}\{2\mathrm{h_i}\},\ \mathrm{i} = 1, \ldots, \mathrm{n}. \end{aligned} \tag{15}$$

The total transformation, $T = T_1 T_2 T_3$, reduces $\boldsymbol{g}$ into an identity matrix and $\boldsymbol{g}_0$ to a diagonal matrix with entries $2h_i$. Note that the last transformation does not change $\boldsymbol{g}$, which has been reduced to the identity matrix in step 2, in accordance with the fact that we are dealing with screws (i.e., $\|\boldsymbol{\omega}\| = 1$). We also note that equation (10) may be interpreted as a pair of quadratic forms over $\Re^n$, namely $\lambda = \dot{\boldsymbol{\theta}}^T \boldsymbol{g} \dot{\boldsymbol{\theta}}$ and $\lambda_0 = \dot{\boldsymbol{\theta}}^T \boldsymbol{g}_0 \dot{\boldsymbol{\theta}}$. It is well known in linear algebra[13] that the two quadratic forms can always be diagonalized *simultaneously*, if one of them is positive definite ($\boldsymbol{g}$ in our case). The simultaneous diagonalization can be done by combining steps 1 and 3 above, and may be shown to be equivalent to the generalized eigen problem

$$\left(\boldsymbol{g}_0 - \frac{\lambda_0}{\lambda}\boldsymbol{g}\right)\dot{\boldsymbol{\theta}}^h = \mathbf{0} \tag{16}$$

From equation (9), we see that the generalized eigenvalues, $\frac{\lambda_0}{\lambda}$, are equal to twice the principal pitches, a result that justifies equation (15). The eigenvectors, $\dot{\boldsymbol{\theta}}^h$, when mapped by $\hat{\boldsymbol{J}}$ leads to *principal screws*, and $\hat{\$}_i^h = \hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}}_i^h$ constitute the $\boldsymbol{h}-$basis. The generalized eigenvectors are mutually orthogonal for distinct generalized eigenvalues, i.e., $\dot{\boldsymbol{\theta}}_i^h \cdot \dot{\boldsymbol{\theta}}_j^h = 0$, if $\hat{\lambda}_i \neq \hat{\lambda}_j$. This leads to the result that $\left\langle \hat{\$}_i^h, \hat{\$}_j^h \right\rangle = 0 + \epsilon 0$. In other words, the principal screws *meet at one point in space orthogonally* if $h_i \neq h_j$ for $i \neq j$ and as proved above, their pitches are extremal. These two observations identify the $\boldsymbol{h}-$basis as the classical principal basis as described in [2, 3]. It may be noted, that as in the $\boldsymbol{\omega}-$basis, the generalized eigenvalues and eigenvectors can be obtained analytically, since we have to solve at most a cubic equation.

3 Analytical expression for principal twists in $\boldsymbol{\omega}-$basis

We now present the important theoretical results obtained in the $\boldsymbol{\omega}-$basis, namely, analytical expressions for the principal twists of multi-DOF rigid-body motion, and partitioning of degrees-of-freedom. We present the results for various degrees-of-freedom.

3.1 One-degree-of-freedom rigid body motion

The simplest case of rigid-body motion is that of one-DOF motion, and the distribution of allowable twists is of the form $\hat{\mathcal{V}} = \hat{\$}_1\dot{\theta}_1$. The single input screw $\hat{\$}_1$ itself may be identified with the principal screw of the system, and transforming to a frame where the $\boldsymbol{X}$ axis is along the screw axis, and the origin is some chosen point on the axis, the principal twist can be written as

$$\hat{\mathcal{V}}_i^* = k(1 + \epsilon h^*)(1, 0, 0)^T \tag{17}$$

where h^* is the pitch of $\hat{\$}$ and $k \in \Re^+$ is the magnitude of the input, assumed to be unity under the unit-speed constraint.

3.2 Two-degrees-of-freedom rigid body motion

For two-DOF motion of a rigid-body, let $\boldsymbol{\theta}(t) = (\theta_1(t), \theta_2(t))^T$ represent the two independent motion parameters. Let $\hat{\$}_i = \boldsymbol{Q}_i + \epsilon(h_i\boldsymbol{Q}_i + \boldsymbol{Q}_{oi})$ represent the ith input screw. The resultant twist can be written as

$$\hat{\mathcal{V}} = \hat{\$}_1\dot{\theta}_1 + \hat{\$}_2\dot{\theta}_2 = \hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}} \tag{18}$$

Following the development in the last section, we obtain the matrix $\hat{\boldsymbol{g}}$, where[3] $\hat{g}_{ij} = c_{ij} + \epsilon((h_i + h_j)c_{ij} - d_{ij}s_{ij})$, $i, j = 1, 2$. In particular, $\hat{g}_{ii} = 1 + \epsilon(2h_i)$, $i = 1, 2$. The dual characteristic equation may be written in its real and dual components as

$$\begin{aligned} \lambda^2 - 2\lambda + s_{12}^2 &= 0 \\ 2(\lambda - 1)\lambda_0 + (h_1 + h_2)(d \sin 2\phi_{12} + 2s_{12}^2 - 2\lambda) &= 0 \end{aligned} \tag{19}$$

The last equation shows that $\lambda_i \neq 0$ iff $s_{12} \neq 0$, and for this condition the two dual eigenvalues are

$$\begin{aligned} \hat{\lambda}_1 &= 2\cos^2\phi_{12}/2(1 + \epsilon(h_1 + h_2 - d_{12}\tan(\phi_{12}/2))) \\ \hat{\lambda}_2 &= 2\sin^2\phi_{12}/2(1 + \epsilon(h_1 + h_2 + d_{12}\cot(\phi_{12}/2))) \end{aligned} \tag{20}$$

The principal magnitude and pitches are given by

$$\begin{aligned} \|\boldsymbol{\omega}_1^*\| &= \sqrt{\lambda_1} = \sqrt{2}\cos\phi_{12}/2 \\ \|\boldsymbol{\omega}_2^*\| &= \sqrt{\lambda_2} = \sqrt{2}\sin\phi_{12}/2 \\ h_1^* &= \frac{\lambda_{01}}{2\lambda_1} = 1/2((h_1 + h_2) - d_{12}\tan(\phi_{12}/2)) \\ h_2^* &= \frac{\lambda_{02}}{2\lambda_2} = 1/2((h_1 + h_2) + d_{12}\cot(\phi_{12}/2)) \end{aligned} \tag{21}$$

The real eigenvectors of $\hat{\boldsymbol{g}}$ are given by $1/\sqrt{2}(1 \pm 1)^T$, and they map to the principal twists as

$$\hat{\mathcal{V}}_{1,2}^* = \frac{1}{\sqrt{2}}(\hat{\$}_1 \pm \hat{\$}_2) \tag{22}$$

It may be verified from the last equation that the principal twists are about two mutually orthogonal, coincident screws.

[3] We use d_{ij} and ϕ_{ij} to denote the distance and angle between the ith and jth screw axes, and c_{ij} and s_{ij} to denote $\cos\phi_{ij}$ and $\sin\phi_{ij}$ respectively.

3.3 Three-degrees-of-freedom rigid body motion

The resultant twist for a rigid body undergoing three-DOF motion, with motion parameters $\boldsymbol{\theta}(t) = (\theta_1(t), \theta_2(t), \theta_3(t))^T$, may be represented by

$$\hat{\boldsymbol{\mathcal{V}}} = \hat{\$}_1\dot{\theta}_1 + \hat{\$}_2\dot{\theta}_2 + \hat{\$}_3\dot{\theta}_3 \tag{23}$$

where $\hat{\$}_i = \boldsymbol{Q}_i + \epsilon(h_i\boldsymbol{Q}_i + \boldsymbol{Q}_{oi})$ is the screw associated with the motion parameter θ_i. We obtain $\hat{\boldsymbol{g}}$ from the inner products, and the real part of the dual characteristic equation of $\hat{\boldsymbol{g}}$ is obtained as

$$\lambda^3 - 3\lambda^2 + (3 - c_{12}^2 - c_{23}^2 - c_{31}^2)\lambda + (c_{12}^2 + c_{23}^2 + c_{31}^2 - 2c_{12}c_{23}c_{31} - 1) = 0 \tag{24}$$

This cubic equation can be solved analytically using Cardan's formula, and using the fact that the roots are real quantities (being the eigenvalues of a symmetric real matrix $\boldsymbol{g}$), the analytical expressions for the roots are obtained as:

$$\lambda_i = 1 + \frac{2\sqrt{3}}{3}\sqrt{c_{12}^2 + c_{23}^2 + c_{31}^2}\cos\left(\frac{\phi + (i-1)2\pi}{3}\right) \tag{25}$$

where $i = 1, 2, 3$, and $\phi \in [0, 2\pi]$ is such that $\sin\phi = (1/27)(c_{12}^2 + c_{23}^2 + c_{31}^2)^3 - c_{12}^2c_{23}^2c_{31}^2$ and $\cos\phi = c_{12}c_{23}c_{31}$. Analogous to the two-degree-of-freedom case, the principal magnitudes and pitches associated with these twists are obtained as $\|\boldsymbol{\omega}_i^*\| = \sqrt{\lambda_i},\ i = 1, 2, 3$, and

$$h_i^* = -\frac{a_2\lambda_i^2 + a_1\lambda_i + a_0}{3\lambda_i^3 - 6\lambda_i^2 + (3 - (c_{12}^2 + c_{23}^2 + c_{31}^2))\lambda_i} \tag{26}$$

Using the notation $H = h_1 + h_2 + h_3$, where $h_i,\ i = 1, 2, 3$ are the pitches of the input screws, we have

$$\begin{aligned} a_2 &= -2H \\ a_1 &= H(2 - c_{12} - c_{23} - c_{31}) + h_1c_{23} + h_2c_{23} + h_3c_{12} \\ a_0 &= H(\cos_{12} + \cos_{23} + \cos_{31} - 4c_{12}c_{23}c_{31}) + 2d_{12}(c_{23}c_{31} - c_{12}) \\ &+ 2d_{31}(c_{12}c_{23} - c_{31}) + 2d_{23}(c_{12}c_{31} - c_{23}) \end{aligned} \tag{27}$$

The ith eigenvector of $\hat{\boldsymbol{g}}$, may be obtained as $\dot{\boldsymbol{\theta}}_i = \left(\frac{c_{12}c_{31}+c_{23}(1+\lambda_i)}{(1+\lambda_i)^2-c_{12}^2}, \frac{c_{12}c_{23}+c_{31}(1+\lambda_i)}{(1+\lambda_i)^2-c_{12}^2}, 1\right)^T$. Normalizing these eigenvectors and writing them as $(l_i, m_i, n_i)^T$, the principal twists are obtained from equation (23) as

$$\hat{\boldsymbol{\mathcal{V}}}_i^* = \hat{\$}_1 l_i + \hat{\$}_2 m_i + \hat{\$}_3 n_i, \quad i = 1, 2, 3 \tag{28}$$

It may be verified that for distinct λ_i, the axes of the principal twists are orthogonal, but they do not coincide in space in general.

3.4 Rigid-body motion with DOF > 3

The general case of n-DOF motion can be considered within the same framework as above. We note that the $rank_{\Re}(\boldsymbol{J}_{\boldsymbol{\omega}}) \leq 3$, and hence $rank_{\Delta}(\hat{\boldsymbol{g}}) \leq 3$, which restricts the characteristic polynomial of $\hat{\boldsymbol{g}}$ to at most a dual cubic. More explicitly, the characteristic equation (40) (see Appendix B) takes the form

$$\hat{\lambda}^{n-3}(\hat{\lambda}^3 + \hat{a}_{n-1}\hat{\lambda}^2 + \hat{a}_{n-2}\hat{\lambda} + \hat{a}_{n-3}) = 0 \tag{29}$$

We conclude from the above that $n-3$ of the eigenvalues are zeros, and the 3 non-zero ones can be computed from the residual cubic equation, once the coefficients are computed from the dual invariants of $\hat{\boldsymbol{g}}$ (see Appendix B). We also note that $a_{n-1} = -n$, as it is the negative of the trace of $\boldsymbol{g}$ and $\hat{\boldsymbol{g}}_{ii} = 1 + \epsilon(2h_i)$. The residual cubic equation, $\lambda^3 - n\lambda^2 + a_{n-2}\lambda + a_{n-3} = 0$, requires the computation of only two coefficients, which are the second and the third invariants of $\boldsymbol{g}$. Thus, by exploiting the algebraic structure of the problem, we ensure an analytic solution for rigid-body motion of arbitrary DOF greater than 3. The 3 eigenvectors corresponding to the non-zero eigenvalues can be computed by the standard method. However, we will also have at least $n-3$ principal twists in the null-space of $\hat{\boldsymbol{J}}$, and to compute them, we first have to find the vectors $\dot{\boldsymbol{\theta}}_i$ in the null space of $\boldsymbol{g}$, by solving the equation

$$\boldsymbol{g}\dot{\boldsymbol{\theta}}_i = \boldsymbol{0} \quad i = 1, \ldots, n-3 \tag{30}$$

where $\boldsymbol{g}$ is a $n \times n$ real symmetric matrix, whose rank is at the most 3. The eigenvectors in the null space can be obtained by row-reducing $\boldsymbol{g}$ to get 3 independent equations, and choose the $n-3$ free variables in each of them suitably. The corresponding principal twist is obtained by the mapping $\hat{\boldsymbol{\mathcal{V}}}_i = \hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}}_i$, and since these twists have only their dual parts, they span the space of pure translational motions of the rigid-body.

3.5 Partitioning of DOF

If $n > 3$, or $rank_{\Re}(\boldsymbol{J}_{\boldsymbol{\omega}}) < min(n, 3)$, one or more of the principal twists will lie in the left null-space of $\hat{\boldsymbol{J}}$. These twists may be computed from equation (5), where $\dot{\boldsymbol{\theta}}_i$ are the eigenvectors corresponding to the vanishing eigenvalues of $\hat{\boldsymbol{g}}$. Expressed as dual vectors, these twists are of the form $\boldsymbol{0} + \epsilon\boldsymbol{v}_i^* = \epsilon\boldsymbol{J}_{\boldsymbol{v}}\dot{\boldsymbol{\theta}}_i, \quad (i = 1, \ldots, n-3)$ for n-DOF motion ($n > 3$). These twists have infinite pitches, and they signify *pure translational motion* of the rigid-body. Rigid-body motion can thus be divided into two parts, namely, one consisting of both rotation and translation (finite-pitch motion), and another consisting of purely translational motion, and independent of the rotational motion of the rigid-body. This decoupling occurs due to the fact that the dual inner product of two pure translational twists is zero, and hence the pure translations lie in the left null-space of $\hat{\boldsymbol{J}}$, which is the orthogonal complement of the column-space. This decoupling *DOF partitioning* allows us to study the rotational and translational modes of rigid-body motion independent of each other, and our *analytical* expressions for the principal twists can now be profitably used for robotic applications where the end-effector motion requirements can be split into these two modes explicitly. We also note here that if $n > 6$, then the number of principal twists in the left null-space of $\hat{\boldsymbol{J}}$ will be more than 3, and their dual parts will be linearly dependent. However, we can always construct an orthogonal basis for subspace of $\Re^3$ spanned by the dual parts, which will give us the distribution of pure translational motions of the rigid body.

4 Analysis of singularities in the $\boldsymbol{\omega}-$basis

In the previous section, we have developed the analytical expressions of the principal twists. For analysis of singularities, we can readily use the analytical expressions to obtain the principal singular directions. In this section, we discuss both the loss and gain kinds of singularities, while noting that the former type is possible only in serial manipulators, and the later in parallel manipulators, while hybrid manipulators can show both types of singularities[14].

4.1 Loss type of singularity

The loss kind of singularity is said to occur when the manipulator end-effector fails to twist about certain screw(s) in spite of full actuation. This results in the loss of one or more degrees-of freedom of the end-effector[15]. In our formulation, we treat the rotational degrees-of-freedom as decoupled from purely translational degrees-of-freedom, and hence the loss may occur in either rotational or translational DOF. We first consider loss of rotational DOF.

The manipulator end-effector has $1, 2$ or 3 rotational degrees-of-freedom depending upon the number of non-zero eigenvalues $\hat{\boldsymbol{g}}$ has at a non-singular configuration. If at a singular configuration, m additional eigenvalues vanish[4], then we say that the manipulator has lost m rotational degrees-of-freedom. It may be noted that the corresponding pitch also vanishes, and hence the corresponding twist reduces to a pure translation in the null-space of $\hat{\boldsymbol{J}}$ at that configuration. We look at the possibilities on a case by case basis.

One-degree-of-freedom: In this case, the principal screw reduces to a null vector, $\mathbf{0} + \epsilon \mathbf{0}$, unless the original DOF was translational (as in a P-joint), in which case there is no loss of rotational DOF possible.

Two-degrees-of-freedom: From the set of equations (19), it can be seen that only one of the $\hat{\lambda}$s ($\hat{\lambda}_2$ in particular) can vanish, under the condition $\sin^2 \phi_{12} = 0$. The other eigenvalue can be obtained from equation(19) as $\hat{\lambda}_1 = 2(1 + \epsilon(h_1 + h_2))$. The two principal twists in equation (22) collapse to $\hat{\boldsymbol{\mathcal{V}}}_1^* = \frac{1}{\sqrt{2}}(\hat{\$}_1 + \hat{\$}_2)$ which gives the resultant rotational DOF in this case, and $\hat{\boldsymbol{\mathcal{V}}}_2^* = \frac{1}{\sqrt{2}}(\hat{\$}_1 - \hat{\$}_2)$, now forms the left null-space of $\hat{\boldsymbol{J}}$, signifying a translatory DOF in addition to the residual rotational DOF.

Three-degrees-of-freedom: In this case, there may be loss of one or two angular degrees-of-freedom, the conditions of the same are found from equation (24) as $c_{12}^2 + c_{23}^2 + c_{31}^2 - 2c_{12}c_{23}c_{31} - 1 = 0$ and $c_{12}^2 + c_{23}^2 + c_{31}^2 - 2c_{12}c_{23}c_{31} - 1 = 0 = (3 - c_{12}^2 - c_{23}^2 - c_{31}^2)$ respectively. As in the case of two-degrees-of-freedom rigid-body motion, the non-zero roots may be computed from equation (24), which reduces to a quadratic and a linear equation in λ in the two cases respectively. The eigenvectors of $\boldsymbol{g}$ can be computed symbolically, and therefrom the principal twists in the column-space and null space of $\hat{\boldsymbol{J}}$ can be obtained using equation (28). It may be noted here that the loss of one or two rotational DOF results in those many principal twists being pushed from the column-space into the left null-space of $\hat{\boldsymbol{J}}$, which has interesting consequences when DOF is greater than 3.

Degrees-of-freedom(n)> 3: The treatment in this case follows exactly the case of three-degrees-of-freedom. We need to consider equation (29) instead of equation (24), and the conditions for loss of one or two rotational DOF are $a_{n-3} = 0$, and $a_{n-3} = 0 = a_{n-2}$ respectively.

The number of pure translational degrees-of-freedom equal the number of linearly independent pure dual vectors in the left null space of $\hat{\boldsymbol{J}}$ and they span the space of pure translational velocities of the rigid body. We write their dual parts as the columns of a $3 \times m$ real matrix, $\mathbf{B}$, and let the rank of $\mathbf{B}$ be $r(r \leq 3)$. At a singularity leading to loss of translational DOF, the rank of $\mathbf{B}$ reduces by 1, 2 or 3. It may be noted that loss of rotational motion also leads to the addition of a column to $\mathbf{B}$, but since the rank of $\mathbf{B}$ is limited to 3, the degeneracy of rotational motion does not lead to an additional translational DOF if rank of $\mathbf{B}$ is already 3.

4.2 Gain type of singularity

A parallel device gains one or more degrees-of-freedom in the configuration space when one of the constraint Jacobians, $\boldsymbol{J}_{\eta\phi}$, loses rank (see Appendix A for derivation of Jacobians for parallel and

[4] m can be either 1 or 2. All the three eigenvalues can vanish only for a purely Cartesian manipulator, whose analysis can be done much more conveniently by looking at its linear velocity distribution in $\Re^3$.

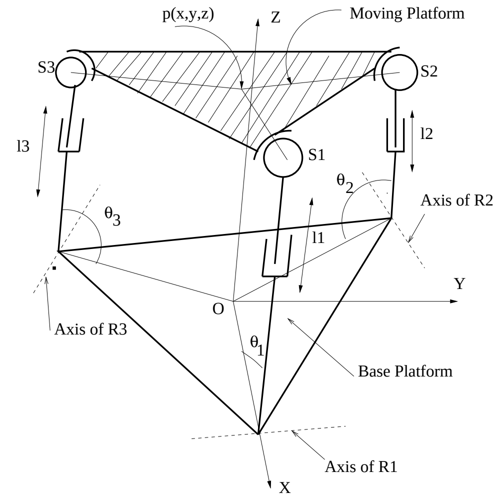

Figure 1: The 3-RPS Parallel Manipulator

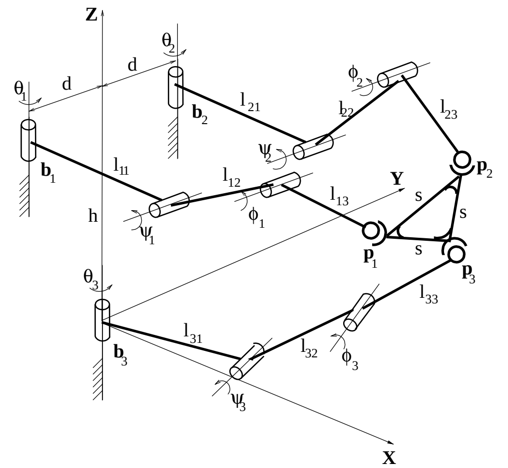

Figure 2: The 6-DOF Hybrid Manipulator

hybrid manipulators), and the number of DOF gain equals the nullity of $\boldsymbol{J}_{\boldsymbol{\eta}\boldsymbol{\phi}}$ (see, for example, [16]). The *gained* passive motions lie in the null-space of $\boldsymbol{J}_{\boldsymbol{\eta}\boldsymbol{\phi}}$, and may be obtained by solving the equation

$$\boldsymbol{J}_{\boldsymbol{\eta}\boldsymbol{\phi}}\dot{\boldsymbol{\phi}}_i = \boldsymbol{0}, \quad i = 1, \ldots, nullity(\boldsymbol{J}_{\boldsymbol{\eta}\boldsymbol{\phi}}) \tag{31}$$

The effect of this gain is that the manipulator end-effector can now twist about one or more screws even with all the actuators locked. These twists are given by

$$\hat{\boldsymbol{\mathcal{V}}}_i = \boldsymbol{J}_{\boldsymbol{\omega}\boldsymbol{\phi}}\dot{\boldsymbol{\phi}}_i + \epsilon \boldsymbol{J}_{\boldsymbol{v}\boldsymbol{\phi}}\dot{\boldsymbol{\phi}}_i \tag{32}$$

We can obtain the *gained screws* $\hat{\boldsymbol{\$}}_i$ by normalizing $\hat{\boldsymbol{\mathcal{V}}}_i$. Any *gained twist* may be written as $\hat{\boldsymbol{\mathcal{V}}}_{\text{gain}} = \sum_{i=1}^{nullity(\boldsymbol{J}_{\boldsymbol{\eta}\boldsymbol{\phi}})} c_i\hat{\boldsymbol{\$}}_i,\ c_i \in \Re$. This equation is comparable with $\hat{\boldsymbol{\mathcal{V}}} = \hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}}$, the gained screws replacing the input screws and the arbitrary coefficients c_i taking the place of $\dot{\theta}_i$. Under a normalization constraint (similar to the unit-speed constraint) $\sum_{i=1}^{nullity(\boldsymbol{J}_{\boldsymbol{\eta}\boldsymbol{\phi}})} c_i^2 = 1$, the principal twists in the space of gained twists can be obtained analytically. This is because, again, we need to solve at the most a cubic equation.

5 Illustrative examples

The above developed theory is illustrated by an examples of a 3-DOF parallel manipulator shown in figure 1 and a 6-DOF hybrid manipulator shown in figure 2. At a non-singular configuration for the 3-DOF parallel manipulator defined by $l_1 = 1$, $l_2 = 2/3$, $l_3 = 3/4$, and corresponding passive variables $\theta_1 = 0.878516$rad, $\theta_2 = 0.905239$rad and $\theta_3 = 0.120906$rad, the dual eigenvalues of $\hat{\boldsymbol{g}}$ are are computed analytically, yielding the numerical values $\hat{\lambda}_1 = 3.92612 + \epsilon(-0.91996)$, $\hat{\lambda}_2 = 1.87034 + \epsilon(0.44710)$, $\hat{\lambda}_3 = 0 + \epsilon(0)$ and the three principal pitches in the $\boldsymbol{\omega}-$basis are given by $h_1 = -0.117159$, $h_2 = 0.119524$, $h_3 = \infty$ respectively. The principal twists, at this configuration, are given by

$$
\begin{aligned}
\hat{\mathcal{V}}_1^{\omega} &= (1.61698, -1.11205, 0.27354)^T + \epsilon(0.59693, 1.14580, -0.55239)^T \\
\hat{\mathcal{V}}_2^{\omega} &= (-0.63533, -1.06544, -0.57580)^T + \epsilon(0.13730, -0.28080, -0.02015)^T \\
\hat{\mathcal{V}}_3^{\omega} &= (0, 0, 0)^T + \epsilon(0, 0, 0.90320)^T
\end{aligned}
$$

The DOF decoupling is apparent in the purely translational nature of the third principal twist. Intuitively, the existence of one pure translation mode can be reasoned from the fact that the rotary joint axes in the base are in a plane and the top platform can be made to translate parallel to the **Z** axis, without any angular motion, by changing the leg lengths. The strength of our approach is that we can analytically capture this *partitioning* of DOF.

The principal twists of the $\boldsymbol{h}-$basis are computed as

$$
\begin{aligned}
\hat{\mathcal{V}}_1^{h} &= (1.10500, -1.35959, 0)^T + \epsilon(0.55638, 0.85062, -0.80373)^T \\
\hat{\mathcal{V}}_2^{h} &= (9.73597, 7.91281, 6.20082)^T \times 10^{-9} + \epsilon(0, 5.25180, -0.90320 \times 10^{9})^T \times 10^{-9} \\
\hat{\mathcal{V}}_3^{h} &= (9.73597, 7.91281, 6.20082)^T \times 10^{-9} + \epsilon(0, 5.25180, 0.90320 \times 10^{9})^T \times 10^{-9}
\end{aligned}
$$

The principal pitches in $\boldsymbol{h}-$basis are computed as $h_1 = -0.17648$, $h_2 = -2.85962 \times 10^7$, $h_3 = 2.85962 \times 10^7$ respectively. It may be noted that $h_3 = -h_2 \to \infty$, and $||\hat{\boldsymbol{\mathcal{V}}}_2^h|| = ||\hat{\boldsymbol{\mathcal{V}}}_3^h|| \to 0$, even as $\boldsymbol{g}$ has rank 2 as seen in the above set of results. By observation, the direction of the pure translation can be obtained by deducting $\hat{\boldsymbol{\mathcal{V}}}_2^h$ from $\hat{\boldsymbol{\mathcal{V}}}_3^h$. It may be noted that we get $(0,0,0)^T + \epsilon(0,0,2 \times 0.9032)$ which is consistent with the translational velocity obtained in $\boldsymbol{\omega}-$basis. The advantage of exact analytical computation, as opposed to numerical computation, is also clearly seen from the values of the principal screws in $\boldsymbol{h}-$basis. One can observe that some entries are $\mathcal{O}(1)$ whereas others are $\mathcal{O}(10^{-9})$ and most numerical computations will round them off to 0. If they are rounded off to zero, then we will get two translatory modes which is incorrect.

The 6-DOF hybrid spatial manipulator, shown in figure 2, models a three-fingered gripper with the contact points modeled as spherical joints. For this manipulator, the active variables are $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \psi_1, \psi_2, \psi_3)^T$, and the passive variable are given by $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^T$. We choose, the link-lengths as $l_1 = 2l_2 = 4l_3 = 1$, $d = 1/2$, $h = \sqrt{3}/2$, and $s = \sqrt{3}/2$. At a non-singular configuration given by $\boldsymbol{\theta} = (0.2, 0.1, 0.3, -1., -1.2, 1)^T$, and $\boldsymbol{\phi}$ given by $(0.3679, 1.4548, 0.8831)^T$, the dual eigenvalues of $\hat{\boldsymbol{g}}$ are computed as $\hat{\lambda}_1 = 0.03 + \epsilon(0.09)$, $\hat{\lambda}_2 = 2.10 + \epsilon(5.45)$ and $\hat{\lambda}_3 = 1496.45 + \epsilon(1070.41)$. The principal pitches are given by

$$h_1^* = 1.37,\ h_2^* = 1.30,\ h_3^* = 0.36,\ h_4^* =\ h_5^* =\ h_6^* = \infty$$

The principal twists in $\boldsymbol{\omega}-$basis, at this configuration, is given by

$$
\begin{aligned}
\hat{\mathcal{V}}_1 &= (14.56, 35.28, 6.24)^T + \epsilon(15.16, 7.43, 8.29)^T \\
\hat{\mathcal{V}}_2 &= (-1.34, 0.52, 0.17)^T + \epsilon(-2.04, -0.03, 0.03)^T \\
\hat{\mathcal{V}}_3 &= (0.01, -0.04, 0.18)^T + \epsilon(-0.36, -0.19, 0.23)^T \\
\hat{\mathcal{V}}_4 &= (0, 0, 0)^T + \epsilon(-0.09, 0.77, 0.11)^T \\
\hat{\mathcal{V}}_5 &= (0, 0, 0)^T + \epsilon(-0.03, -0.49, 0.06)^T \\
\hat{\mathcal{V}}_6 &= (0, 0, 0)^T + \epsilon(0.19, -0.07, 0.27)^T
\end{aligned}
$$

We consider the singular configuration where all the three fingers are fully stretched[14]. The configuration is defined by $\boldsymbol{\theta} = (0.0500, -0.0500, 0, -1.0998, -1.0998, 1.0026)^T$ and $\boldsymbol{\phi} = (0,0,0))^T$.

We expect a loss of three degrees-of-freedom since all three fingers are in singular configuration, and accordingly we find that the pure dual principal twists vanish identically, signifying the loss of three translational degrees of freedom. The other three principal twists are given as

$$\begin{aligned}
\hat{\mathcal{V}}_1 &= (-1.79, -27.73, 0.05)^T + \epsilon(12.01, -0.01, -8.63)^T \\
\hat{\mathcal{V}}_2 &= (12.25, -0.79, -0.36)^T + \epsilon(1.75, 0.04, -1.26)^T \\
\hat{\mathcal{V}}_3 &= (0.0001, 0, 0.0050)^T + \epsilon(-0.00, -0.76, -0.00)^T
\end{aligned}$$

For a configuration defined by $\boldsymbol{\theta} = (0.0554, -0.0544, -0.8119, -0.8199, 0, 1.5708)^T$, $\boldsymbol{\phi} = (-1.3300, -1.3300, 0.7854)^T$, there is a gain of a single DOF, and the corresponding gained passive motion in the null-space of $\boldsymbol{J}_{\eta\phi}$ is obtained as $(0, 0, 1)^T$, indicating that ϕ_3 has an instantaneous variation even with actuators locked. The gained twist is $(0, 1/3, 0)^T + \epsilon(-1/12, 0, 0)^T$ which indicates angular velocity along Y direction and linear velocity in the $X - Z$ plane. In this example too, the analytical computations yield exact directions and not numerically corrupted values.

6 Conclusion

In this paper, we have presented a formal algebraic framework for analysis of multi-DOF rigid body motions which is applied towards analysis of parallel and hybrid manipulators. The development and the results presented in this paper are heavily based on the use and properties of dual numbers, vectors and matrices. The main contributions are a) analytical expressions for principal twists for arbitrary multi-DOF rigid body motions, b) the concept of DOF partioning, and c) analytical identification of gained or lost twists at singular configurations of a manipulator.

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A Derivation of the Dual Jacobian of Parallel and Hybrid Manipulators

In parallel manipulators, closed loop mechanisms, and hybrid manipulators, in addition to the actuated joints, we have one or more passive joints. A parallel device with m passive variables has m independent constraint equations denoted by

$$\boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{\phi}) = \mathbf{0} \tag{33}$$

where $\boldsymbol{\eta}$ is a m-vector, $\boldsymbol{\theta}$, and $\boldsymbol{\phi}$, are n-and m-vectors denoting the actuated and passive joint variables respectively. Differentiating this equation with respect to time and rearranging[15], we get

$$\boldsymbol{J}_{\eta\theta}\dot{\boldsymbol{\theta}} + \boldsymbol{J}_{\eta\phi}\dot{\boldsymbol{\phi}} = \mathbf{0} \tag{34}$$

At a non-singular configuration, $\boldsymbol{J}_{\eta\phi}$ is invertible, and we can obtain the *passive joint rates* as

$$\dot{\boldsymbol{\phi}} = -\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta}\dot{\boldsymbol{\theta}} \tag{35}$$

For parallel and hybrid manipulators, equation (5) may be written in terms of the Jacobians corresponding to the active and passive variables as

$$\hat{\boldsymbol{\mathcal{V}}} = \boldsymbol{J}_{\omega\theta}\dot{\boldsymbol{\theta}} + \boldsymbol{J}_{\omega\phi}\dot{\boldsymbol{\phi}} + \epsilon(\boldsymbol{J}_{v\theta}\dot{\boldsymbol{\theta}} + \boldsymbol{J}_{v\phi}\dot{\boldsymbol{\phi}}) \tag{36}$$

Eliminating $\dot{\boldsymbol{\phi}}$ using equation(35), we get

$$\begin{aligned}\hat{\boldsymbol{\mathcal{V}}} &= (\boldsymbol{J}_{\omega\theta} - \boldsymbol{J}_{\omega\phi}\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta})\dot{\boldsymbol{\theta}} + \epsilon(\boldsymbol{J}_{v\theta} - \boldsymbol{J}_{v\phi}\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta})\dot{\boldsymbol{\theta}} \\ &= \hat{\boldsymbol{J}}_{\mathrm{eq}}\dot{\boldsymbol{\theta}}\end{aligned} \tag{37}$$

where the dual Jacobian can be written as $\hat{\boldsymbol{J}}_{\mathrm{eq}} = (\boldsymbol{J}_{\omega\theta} - \boldsymbol{J}_{\omega\phi}\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta}) + \epsilon(\boldsymbol{J}_{v\theta} - \boldsymbol{J}_{v\phi}\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta})$, and its columns may be considered as *equivalent input screws.*

B Dual Eigen Problem

The general eigen problem of a square dual matrix $\hat{\boldsymbol{A}} = \boldsymbol{A} + \epsilon \boldsymbol{A}_0$, $\boldsymbol{A}, \boldsymbol{A}_0 \in \Re^{n\times n}$, may be written as

$$\hat{\boldsymbol{A}}\hat{\boldsymbol{x}} = \hat{\lambda}\hat{\boldsymbol{x}} \tag{38}$$

However, here we give the solution for the only case encountered in the paper, i.e., $\hat{\boldsymbol{A}}$ is symmetric, and $\hat{\boldsymbol{x}} = \boldsymbol{x} \in \Re^n$. The real part of the eigen problem in equation (38) is exactly the eigen problem of the real part, $\boldsymbol{A}$:

$$(\boldsymbol{A} - \lambda I)\boldsymbol{x} \quad = \quad \boldsymbol{0} \tag{39}$$

We can compute λ, $\boldsymbol{x}$ from it using the usual techniques. To compute λ_0, the dual part of $\hat{\lambda}$, we equate the determinant $\det(\hat{\boldsymbol{A}} - \hat{\lambda} I)$ to $0 + \epsilon 0$, and obtain the dual characteristic polynomial of the form:

$$\sum_{r=0}^{n} \hat{a}_r \hat{\lambda}^r = \sum_{r=0}^{n} (a_r + \epsilon a_{r0})(\lambda^r + \epsilon r \lambda^{r-1}\lambda_0) = 0 \tag{40}$$

Equating the real and dual parts of the above equation to zero separately, we get

$$\begin{aligned} \sum_{r=0}^{n} a_r \lambda^r &= 0 \\ \sum_{r=1}^{n} a_r r \lambda^{r-1}\lambda_0 + \sum_{r=0}^{n} a_{r0}\lambda^r &= 0, \quad \hat{a}_n = 1 \end{aligned} \tag{41}$$

Solution of the first of equations(41) gives, in general, n values of λ, and for each of these values, we can solve for the corresponding λ_0 *uniquely* from the second. In particular, when λs are unique, λ_0 is given in terms of λ as

$$\lambda_0 = -\frac{\sum_{r=0}^{n} a_{r0}\lambda^r}{\sum_{r=1}^{n} a_r r \lambda^{r-1}} \qquad (\lambda \neq 0) \tag{42}$$

It may be noted here that the construction of the characteristic polynomial by expansion of the determinant of $(\hat{\boldsymbol{A}} - \hat{\lambda} I)$ requires expensive symbolic computation. Alternatively, we can construct the polynomial by explicitly computing the invariants of $\hat{\boldsymbol{A}}$, taking advantage of the principle of *permanence of identities*. The formula required for computation of the invariants is obtained from the matrix-form of Newton's identities:

$$\hat{\mathcal{I}}_k = \frac{(-1)^{k+1}}{k}\left(\operatorname{tr}\left(\hat{\boldsymbol{A}}^k\right) + \sum_{i=1}^{k-1}(-1)^i \hat{\mathcal{I}}_i \operatorname{tr}\left(\hat{\boldsymbol{A}}^{k-i}\right)\right), \; \hat{\mathcal{I}}_1 = \operatorname{tr}\hat{\boldsymbol{A}}$$

where $k = 1, \ldots, n$. The coefficients, $\hat{a}_r$, are obtained from the dual invariants as $\hat{a}_r = (-1)^{n-r}\hat{\mathcal{I}}_{n-r}, r = 1, \ldots, n-1$, while $\hat{a}_n = 1$, since characteristic polynomials are always monic.