Analysis of configuration space singularities of closed-loop mechanisms and parallel manipulators

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Abstract

A parallel manipulator or a closed-loop mechanism may gain or lose one or more degree-of-freedom at a singular point, and in this paper, we study the singularity associated with the gain of one or more degrees-of-freedom. We analyze the constraint forces associated with the kinematic constraints inherent in a closed-loop mechanism or a parallel manipulator, and characterize the gain singularities from the degeneracy of these constraint forces. Several special phenomena associated with gain singularity, such as locking of the actuators have been studied, and analytical criteria for these have been derived. We also present the necessary condition for finite self motion and finite dwell of the passive links by analyzing second-order properties of the constraint equations. The results are illustrated with the help of several closed-loop mechanisms.

1 Introduction

Analysis and evaluation of singularities play an important role in several aspects of mechanisms and robotics including design, trajectory planning, and control. A great deal of research has been done on the singularity analysis of both serial and parallel manipulators and closed-loop mechanisms. The study of serial manipulator configurations resulting in singularities have been done by several researchers (see, for example, [1, 2, 3, 4, 5]). A serial manipulator is said to be in a singular configuration when the manipulator Jacobian matrix loses rank, and in this configuration a serial manipulator loses one or more degree-of-freedom. A parallel manipulator or a closed-loop mechanism, on the other hand, may either *lose* or *gain* one or more degrees-of-freedom at a singular configuration, and this loss or gain has been attributed to the degeneracy of two different Jacobian matrices originating from the time derivative of the input-output equation of the manipulator or closed-loop mechanism[6, 7].

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Gosselin and Angeles[6] also describe another type of singularity, namely *architectural* singularity, which occurs when both the matrices are degenerate for certain special geometries of the closed-loop mechanisms. In such situations, the parallel manipulator or the closed-loop mechanism can exhibit *finite* self-motion or dwell(see also [8]). In addition to the study of singularities in terms of loss or gain of velocities, several researchers have also studied singularities in terms of statics(see, for example, [9, 10, 11, 12]), and have made use of the force transformation matrix. They have obtained conditions under which a parallel manipulator or a closed-loop mechanism cannot withstand external forces or torques in certain directions.

In contrast to serial manipulators, obtaining the singular configurations in parallel manipulators and closed-loop mechanisms is difficult, and very few general results are available. Hunt *et al.*[13] have shown, through screw theory, that a fully in-parallel device can only gain one or more degree-of-freedom, whereas a fully in-series device can only lose one or more degree-of-freedom. A hybrid parallel manipulator may gain or lose one or more degrees-of-freedom. General configurations for singularity in restricted classes of platform type closed-loop mechanisms containing sphere-sphere link have been obtained by Basu and Ghosal[14].

Most of the approaches towards singularity analysis, in the references mentioned above, involve use of linear algebra based techniques, and link the singularities to the rank-deficiency of certain matrices[6]. Merlet[9], has obtained singular configurations of parallel manipulators using Grassman geometry. Researchers have also used techniques from differential geometry to study the local properties of the configuration manifold of the closed-loop mechanisms and parallel manipulators, and identified singularities with the degeneracy of the tangent-space of the configuration manifold[8]. In[15], the authors have used the concept of a metric in the tangent space of the point trajectory of the output link, and studied singularities in the Cartesian space from the degeneracy of the velocity ellipsoid associated with the metric.

A key difference between serial and parallel manipulators(and closed-loop mechanisms) is that in parallel manipulators, there exist *holonomic constraints* in the form of *loop-closure* equations. A parallel manipulator described by n configuration variables with m loop-closure constraint equations will have $n - m$ degrees-of-freedom – only $n - m$ of the configuration variables can be actuated and m of them are passive. It is well known, that associated with the holonomic constraints, there are constraint forces *normal* to the configuration manifold, which do not do any work. In this paper, the focus is on the constraint forces associated with the holonomic constraints and its analysis. This is different from the concept of analyzing the loss and gain of velocities, in the *tangent* space of the configuration manifold, done by most researchers. One of the main contributions of this paper, obtained as a result of analyzing the constraint equations, is that we are able to derive the *necessary analytical criteria* for finite self-motion and finite dwell of links associated with the passive joints of a closed-loop mechanism or a parallel manipulator.

The paper is organized as follows: In section 2, we present the mathematical approach for the analysis of constraint forces and configuration-space singularities of parallel manipulators and closed-loop mechanisms. In section 3, we present the derivation of the necessary criteria for finite self-motion, finite dwell of the passive links, and also discuss criteria of locking of the actuators. In section 4, we illustrate our theoretical results with the help of a planar five-bar, six-bar, a three-degree-of-freedom planar multi-loop mechanism and a six-degree-of-freedom spatial parallel manipulator. Finally, in section 5, we present the conclusions.

2 Singularities in the Configuration Space

2.1 Configuration Space of a Closed Loop Mechanism

The configuration space of a parallel manipulator or closed loop mechanism is the space of the joint variables of the manipulator. In a parallel manipulator having n joints, not all of the joints are actuated and m ($m < n$) of them may be passive. The degree-of-freedom of such a manipulator is $n-m$, and the joint variable, $\mathbf{q} = (q_1, q_2, .., q_n)^T$ satisfies the constraint equations of the form

$$\boldsymbol{\eta}(q_1, ..., q_n) = \mathbf{0} \tag{1}$$

where $\boldsymbol{\eta}(\cdot) = 0$ denotes the m independent constraint functions[1], $\eta_i(\cdot) = 0, i = 1, 2, .., m$. Such constraints arise from the *loop-closure* equations of the closed loop mechanism, and they depend on the architecture and geometry of the mechanism. The configuration space $\mathcal{C}$, is therefore given by $\mathcal{C} = \{\mathbf{q} | \boldsymbol{\eta}(\mathbf{q}) = \mathbf{0}\}$ where $\mathcal{C}$ is a $n-m$ dimensional manifold embedded in an ambient space $\Re^p \times S^{(n-p)}$, and p is the number of prismatic joints and $n-p$ joints are rotary[2]. Each point of $\mathcal{C}$ corresponds to a configuration of the mechanism, and the set of equations, $\boldsymbol{\eta}(q_1, ..., q_n) = \mathbf{0}$, in general represent m constraint hyper-surfaces in $\mathcal{C}$.

[1] In this paper, we restrict ourselves to non-redundant manipulators and closed-loop mechanisms, i.e., $(n-m) \leq 3$ when a point trajectory is of interest and $(n-m) \leq 6$ if the translation and orientation of a rigid body are both of interest.

[2] We consider all joints to have single degree-of-freedom. A joint with more than one-degree-of-freedom, such as a cylindrical joint may be thought of as a combination of a prismatic and a rotary joint.

Differentiating the m constraint equations (1) with respect to time, t, we get

$$\frac{\partial \boldsymbol{\eta}}{\partial t} + \sum_{i=1}^{m} \frac{\partial \boldsymbol{\eta}}{\partial q_i} \dot{q}_i = \mathbf{0} \tag{2}$$

In closed-loop mechanisms and parallel manipulators, the constraint equations, typically, have no explicit dependence on time, and hence we can write

$$\sum_{i=1}^{m} \frac{\partial \boldsymbol{\eta}}{\partial q_i} \dot{q}_i = \mathbf{0} \tag{3}$$

The above equation may be written in matrix form as

$$[\mathbf{N}]\dot{\mathbf{q}} = \mathbf{0} \tag{4}$$

where $\dot{\mathbf{q}}$ denotes the time derivative of the configuration variable, and is given by the vector $(\dot{q}_1, \dot{q}_2, ..., \dot{q}_n)$. The matrix $[\mathbf{N}]$ is given by

$$[\mathbf{N}] = \left[\frac{\partial \eta_i}{\partial q_j}\right] \tag{5}$$

which can also be written as $[\mathbf{N}] = (\mathbf{N}_1^T, \mathbf{N}_2^T, ..., \mathbf{N}_m^T)^T$, where N_i, the i^{th} row vector of $[\mathbf{N}]$, is the *gradient vector* to the i^{th} constraint hyper-surface in $\mathcal{C}$. Equation(4) implies that the motion of the system in the configuration space is orthogonal to the *normals* to all the constraint surfaces, i.e., the configurational motion lies in the tangent space $T_{\mathbf{q}}\mathcal{C} \quad \forall \mathbf{q} \in \mathcal{C}$.

2.2 Geometric Description of the Constraint Forces and Singularity

It is well known that associated with the m constraints, there are m constraint forces, denoted by the m-vector $\mathbf{F}_c$, which do not do any work. This leads to the observation that $\mathbf{F}_c$ is orthogonal to $T_{\mathbf{q}}\mathcal{C}$, which may be expressed as

$$\mathbf{F}_c^T \dot{\mathbf{q}} = \mathbf{0} \tag{6}$$

By comparing equations (4) and (6), we get

$$\mathbf{F}_c = \sum_{i=1}^{m} \lambda_i \mathbf{N}_i = [\mathbf{N}]^T \boldsymbol{\lambda} \tag{7}$$

where λ_i's are the components of a *non-null* m-vector, $\boldsymbol{\lambda}$. Equation(7) shows that $\mathbf{F}_c$ is a linear combination of the *gradients* to the constraint surfaces, $\mathbf{N}_i$, while the λ_i's are the corresponding weights. At a *generic* point, the vectors $\mathbf{N}_i$ form a linearly independent set, and the matrix $[\mathbf{N}]$ is of full rank, i.e., $rank([\mathbf{N}]) = m$. We consider a point, where $rank([\mathbf{N}]) < m$. At such a point, the rows of the matrix $[\mathbf{N}]$ become linearly dependent, and one or more of the gradient vectors, $\mathbf{N}_i$, may be expressed as linear combination(s) of the others. This implies that the contribution to $\mathbf{F}_c$ from such constraint(s), given by $\lambda_k \mathbf{N}_k$ for some k, gets algebraically added to the contribution from the others, and equivalently, we can say that at that point, one or more kinematic constraints are no longer active. This degeneracy of the constraint forces is reflected in the appearance of the null-space of the matrix$[\mathbf{N}]$, which is orthogonal to $\mathbf{F}_c$. As a consequence, any velocity $\dot{\mathbf{q}}$ in $T_{\mathbf{q}}\mathcal{C}$, belonging to the null-space of $[\mathbf{N}]$ will satisfy equation(4), even if it corresponds to zero motion in the actuators. This leads to the possibility of non-zero velocity in the passive joints with actuators locked, i.e., a gain in degree-of-freedom of the mechanism at such a singular point. We note that the degeneracy of the constraints is only *local* at a point $\mathbf{q} \in \mathcal{C}$, and hence the gained passive velocity is *instantaneous*. We analyze the nature of the gained velocity in detail in the following discussion.

2.3 Gained Velocity with Actuators Locked

To analyze the *gained passive velocity*, we consider all the actuators to be locked, i.e., $\dot{\theta} = \mathbf{0}$. Equation (4) may be decomposed into active and passive parts as

$$[\mathbf{K}]\dot{\boldsymbol{\theta}} + [\mathbf{K}^*]\dot{\boldsymbol{\phi}} = \mathbf{0} \tag{8}$$

where we denote the $(n-m)$ active variables by the vector $\boldsymbol{\theta}$ and the m passive variables by the vector $\boldsymbol{\phi}$, such that $\mathbf{q} = (\boldsymbol{\theta}^T, \boldsymbol{\phi}^T)^T$, and $\dot{\boldsymbol{\theta}}$, $\dot{\boldsymbol{\phi}}$ are the time-derivatives of $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ respectively. The columns of $[\mathbf{K}]$ and $[\mathbf{K}^*]$ contain the partial derivatives of $\boldsymbol{\eta}$ with respect to $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ respectively. With actuators locked($\dot{\boldsymbol{\theta}} = \mathbf{0}$), equation(8) reduces to $[\mathbf{K}^*]\dot{\boldsymbol{\phi}} = \mathbf{0}$, and effectively, $[\mathbf{K}^*]$ plays the role of $[\mathbf{N}]$ – this may be shown by reformulating the constraint equations with the actuators locked. All the above observations made in terms of $[\mathbf{N}]$ are equally valid for $[\mathbf{K}^*]$, and indeed, the rows of $[\mathbf{K}^*]$ may be visualized as the projections of $\mathbf{N}_i$ onto the passive subspace of $T_{\mathbf{q}}\mathcal{C}$. Hence the gain of degree-of-freedom in the configuration space requires the rows of $[\mathbf{K}^*]$ to become linearly dependent, i.e., *singularity criterion* is given by

$$\det{[\mathbf{K}^*]} = 0 \tag{9}$$

It may be noted that $[\mathbf{K}^*]$ is always a $m \times m$ matrix, as there are always m constraint equations and m passive variables $\boldsymbol{\phi}$ in a $n-m$ degree-of-freedom closed-loop mechanism or parallel manipulator.

The singularity condition, $\det[\mathbf{K}^*] = 0$, can also be arrived from linear algebra: from equation(8), if $\dot{\boldsymbol{\theta}} = \mathbf{0}$ and $\det[\mathbf{K}^*] \neq 0$, then $\dot{\boldsymbol{\phi}} = \mathbf{0}$. If $\det[\mathbf{K}^*] = 0$ and $\dot{\boldsymbol{\theta}} = \mathbf{0}$, then there exists $\dot{\boldsymbol{\phi}}_n \in Null([\mathbf{K}^*])$, where $Null([\mathbf{K}^*]) = \{\mathbf{x}|\ [\mathbf{K}^*]\mathbf{x} = 0,\ |\mathbf{x}| \neq 0\}$. Hence the mechanism is in a configuration corresponding to gain of degree(s)-of-freedom, and the gained passive velocity lie(s) in the null-space of $[\mathbf{K}^*]$.

The velocity in the configuration space, given by $\dot{\mathbf{q}} = (\mathbf{0}^T, \dot{\boldsymbol{\phi}}_n^T)^T$ is in the null-space of $[\mathbf{N}]$. As the constraint forces are restricted to the row-space of $[\mathbf{N}]$, and the gained velocity is in the null-space of $[\mathbf{N}]$, the orthogonality between the constraint force and gained velocity is maintained [3].

2.4 Gain Singularity and Locking

In the above discussion, we have assumed the actuators to be locked in order to analyze the *gain* of one or more degrees-of-freedom. We shall now study whether the converse is true, i.e., if gain singularity implies locking of the actuators or *actuator singularity*, as termed by some authors[8]. For the purpose of analysis, we decompose the constraint force, $\mathbf{F}_c$, into two parts, as

$$\mathbf{F}_c = (\mathbf{F_{c\theta}}^T, \mathbf{F_{c\phi}}^T)^T \tag{10}$$

where $\mathbf{F_{c\theta}}$ is the *active* part(associated with the actuated joints $\boldsymbol{\theta}$), and $\mathbf{F_{c\phi}}$ is the *passive* part(associated with the passive joints $\boldsymbol{\phi}$). We can obtain explicit expressions for these quantities from equation(6) as follows:

$$\begin{pmatrix} \mathbf{F_{c\theta}} \\ \mathbf{F_{c\phi}} \end{pmatrix} = \begin{pmatrix} [\mathbf{K}]^T\boldsymbol{\lambda} \\ [\mathbf{K}^*]^T\boldsymbol{\lambda} \end{pmatrix} \tag{11}$$

Pre-multiplying the equation(6) with $[\mathbf{N}]$, we get

$$[\mathbf{N}]\mathbf{F}_c = [\mathbf{N}][\mathbf{N}]^T\boldsymbol{\lambda} \tag{12}$$

Writing $\mathbf{F}_c$ and $[\mathbf{N}]$ explicitly in terms of their component matrices, we get

$$[\mathbf{K}]\mathbf{F_{c\theta}} + [\mathbf{K}^*]\mathbf{F_{c\phi}} = [\mathbf{g_f}]\boldsymbol{\lambda} \tag{13}$$

where $[\mathbf{g_f}] = [\mathbf{N}][\mathbf{N}]^T$ is a symmetric square matrix. From the last equation, we can extract $\mathbf{F_{c\phi}}$ as

$$\mathbf{F_{c\phi}} = [\mathbf{K}^*]^{-1}([\mathbf{g_f}]\boldsymbol{\lambda} - [\mathbf{K}]\mathbf{F_{c\theta}}) \tag{14}$$

[3] We know from linear-algebra that the null-space of any matrix is the orthogonal complement of its row-space[16].

The vector in the parenthesis *cannot* be a null-vector, as then equation(13) would require that $\mathbf{F_{c\phi}}$ lie in the *null space* of $[\mathbf{K}^*]$, while we know from equation(11), that $\mathbf{F_{c\phi}}$ belongs to the *row-space* of $[\mathbf{K}^*]$. Hence equation(14) will yield finite $\mathbf{F_{c\phi}}$ *iff* $\det[\mathbf{K}^*] \neq 0$. In other words, at a gain-singularity, since $\det[\mathbf{K}^*] = 0$, $\mathbf{F_{c\phi}}$ will be infinite. Since constraint forces arise out of the *internal forces*, the mechanism will be able to withstand infinite internal forces as an ordinary structure, which shows that it has all the actuators in a locked situation. The above justifies our analysis of the gained passive velocity keeping the actuators fixed, and establishes the new result that *the actuator singularity is a manifestation of the gain singularity.*

3 Second-order Analysis of Constraint Equations

In the previous section, we have analyzed the matrices, $[\mathbf{N}]$, $[\mathbf{K}]$ and $[\mathbf{K}^*]$ which arise from the first derivative of the constraint equations, and related their degeneracy to the *instantaneous* gain in degree-of-freedom at an isolated singular point of $\mathcal{C}$. In this section, we analyze the second-order properties related to the derivatives of $\det[\mathbf{K}^*]$ and elements of $[\mathbf{N}]$. We show that the second-order analysis leads to analytical criteria for finite self motion(FSM) and finite dwell(FD).

3.1 Gain Singularity and Finite Self Motion

Finite self motion of a mechanism refers to finite movement of the passive parts of the mechanism, with the actuators held fixed. Finite self motion clearly requires instantaneous gain-singularity. However, in addition, the gain-singularity is maintained over a *finite span* of motion of the corresponding passive parts. It is known that FSM imposes restrictions on the architecture of the mechanism[6], and in the following discussion, we show how these architectural requirements may be obtained from the analysis of the constraint equations along with the singularity criteria.

As FSM requires existence of gain singularity, a *necessary* condition is that $rank([\mathbf{K}^*]) < m$. The number of passive joints, that can have nonzero velocity independent of the input, is given by the nullity of $[\mathbf{K}^*]$, denoted by $\mathcal{N}(K^*)$. For the purpose of analysis, we partition $\boldsymbol{\phi}$ in two parts, namely $\boldsymbol{\phi}^I$ of dimension $\mathcal{N}([K^*])$ and $\boldsymbol{\phi}^D$ of dimension $m - \mathcal{N}([K^*])$, which denote the *independent* and *dependent* parts of $\boldsymbol{\phi}$ respectively. We consider the case when $\mathcal{N}(K^*) = 1$ and present the results in the form of the following theorem:

Theorem 3.1 *The necessary condition for FSM in a mechanism with gain of one degree-of-freedom is* $\frac{d}{d\phi^I}\det[\mathbf{K}^*] = 0$ *over the span of* ϕ^I *where* $\det[\mathbf{K}^*] = 0$.

Proof: The mechanism will have a degree-of-freedom in the form of a motion of ϕ^I iff the constraint equations are *at least locally* independent of the variable ϕ^I. We equate the total derivative of $\boldsymbol{\eta}$ to zero to obtain the equation

$$\frac{d\boldsymbol{\eta}}{d\phi^I} = \sum_{k=1}^{m} \frac{\partial \boldsymbol{\eta}}{\partial \phi_k}\frac{d\phi_k}{d\phi^I} = 0 \tag{15}$$

Note that $\frac{d\boldsymbol{\eta}}{d\phi_k}$ gives the k^{th} column of $[\mathbf{K}^*]$, and equation (15) may also be written as

$$[\mathbf{K}^*]\left(\frac{d\phi_1}{d\phi^I},\ \frac{d\phi_2}{d\phi^I},\ \ldots,\ \frac{d\phi_{j-1}}{d\phi^I},\ 1,\ \frac{d\phi_{j+1}}{d\phi^I},\ \ldots,\ \frac{d\phi_m}{d\phi^I}\right)^T = \mathbf{0} \tag{16}$$

where ϕ_j has been chosen as ϕ^I for simplicity. The above equation can have a non-trivial solution *iff* $\det[\mathbf{K}^*] = 0$, which is the same as the criteria for instantaneous gain of degree-of-freedom. Further, FSM implies instantaneous gain of degree-of-freedom over a finite interval of ϕ^I. This implies $\frac{d}{d\phi^I}(\det[\mathbf{K}^*]) = 0$ over the interval of ϕ^I where we have instantaneous gain of one-degree-of-freedom. ■

At a non-singular point, we have $\frac{d\phi_k}{d\phi^I} = \delta_{jk}$ (δ_{jk} denotes the Krönecker delta), since the passive variables do not have any explicit dependence on each other. At a singular point, $\boldsymbol{\phi}^D$ is dependent on $\boldsymbol{\phi}^I$, and $\frac{d\phi_k}{d\phi^I} \neq 0$. Equation(16) gives m linear equations in $m-1$ derivatives of the form $\frac{d\phi_k}{d\phi^I}$. We also have the following second-order relationship:

$$\frac{d}{d\phi^I}(\det[\mathbf{K}^*]) = 0 \tag{17}$$

which is linear in $\frac{d\phi_k}{d\phi^I}$'s. Using this extra equation, we can deduce one architectural requirement of the mechanism which yields FSM in a singular configuration. Note that these equations involve one or more of the passive variables ϕ^I, and corresponding link-parameters, hence the criteria for FSM involves these parameters directly. Substituting these relationships and the singularity criteria in the original constraint equation(1), we obtain the configurational and architectural requirements on the other parts of the mechanism which allows FSM, but are not involved in the motion.

3.2 Dwell of Passive Links : Instantaneous and Finite

Dwell of a passive link refers to a situation when the link is at rest for *instantaneous* or *finite* motion of the actuators. While instantaneous dwell (ID) is the more frequent of the two and can occur for *general* architecture of the mechanism, finite dwell (FD) requires some constraints on the architecture, as in the case of FSM. In the following discussion, we analyse the criteria for both ID and FD in a mechanism. We start by proving two theorems:

Theorem 3.2 *The link associated with the i^{th} joint of a mechanism dwells instantaneously with arbitrary input $\dot{\boldsymbol{\theta}}$, if the i^{th} row of the matrix $[\mathbf{K}^*]^{-\mathbf{1}}(-[\mathbf{K}])$, denoted by $\mathbf{R}_i$, becomes null.*

Proof: Velocity of the i^{th} passive joint, $\dot{\phi}_i$, will be zero, if it is not *kinematically influenced* by any of the actuated joints. Mathematically, this implies that $\frac{\partial \phi_i}{\partial \theta_j} = 0 \; j = 1, n - m$. Assuming the passive variables ϕ_i's as explicit functions of $\boldsymbol{\theta}$, we can write

$$\dot{\phi}_i = \frac{\partial \phi_i}{\partial \theta_j} \dot{\theta}_j \qquad j = 1, ..., n - m \tag{18}$$

Clearly, ID of the link associated with joint i will occur with arbitrary input $\dot{\boldsymbol{\theta}}$, if we have $\frac{\partial \phi_i}{\partial \theta_j} = 0, \; j = 1, n - m$. From equation (8), $\dot{\boldsymbol{\phi}} = [\mathbf{K}^*]^{-1}(-[\mathbf{K}])\dot{\boldsymbol{\theta}}$. Comparing with equation (18), we note that the element (i, j) of $[\mathbf{K}^*]^{-\mathbf{1}}(-[\mathbf{K}])$ may be written as

$$[[\mathbf{K}^*]^{-\mathbf{1}}(-[\mathbf{K}])]_{ij} = \frac{\partial \phi_i}{\partial \theta_j} \tag{19}$$

$\frac{\partial \phi_i}{\partial \theta_j}$, for $j = 1, n - m$ gives the i^{th} row of $[\mathbf{K}^*]^{-\mathbf{1}}(-[\mathbf{K}])$, hence the link associated with joint i will have ID if the i^{th} row of $[\mathbf{K}^*]^{-\mathbf{1}}(-[\mathbf{K}])$ is null.■

Theorem 3.3 *The link associated with the i^{th} joint of a mechanism dwells finitely with arbitrary input $\dot{\boldsymbol{\theta}}$, if*

1. $\mathbf{R}_i = \mathbf{0}$
2. $\frac{d\boldsymbol{\eta}}{d\boldsymbol{\theta}} = \mathbf{0}$
3. $\frac{d\mathbf{R}_i}{d\boldsymbol{\theta}} = \mathbf{0}$

Proof: Clearly FD requires ID, hence we must have $\mathbf{R}_i = \mathbf{0}$. Further, the active joint has finite motion while some passive link dwells, hence we must have the constraint equations independent of the relevant active variable. Finally, the passive link dwell over a finite span of the motion of the active link also implies that the ID criteria is independent of the motion of the active variable in the FD span. ■

We now investigate the structure of the above equations, and extract the architectural requirements for FD from them. Expanding $\frac{d\boldsymbol{\eta}}{d\boldsymbol{\theta}}$ by chain rule of derivatives, we get

$$\frac{d\boldsymbol{\eta}}{d\theta_i} = \frac{\partial\boldsymbol{\eta}}{\partial\theta_i} + \sum_{k=1}^{m} \frac{\partial\boldsymbol{\eta}}{\partial\phi_k}\frac{d\phi_k}{d\theta_i} = 0 \qquad i = 1, n-m \tag{20}$$

The above gives a set of $m \times (n-m)$ scalar equations in an equal number of total derivatives of the form $\frac{d\phi_k}{d\theta_i}$. Similarly, if we have n_d numbers of passive links in finite dwell, we obtain $n_d \times (n-m)^2$ scalar equations in $\frac{d\phi_k}{d\theta_i}$'s from $\frac{d\mathbf{R}_i}{d\boldsymbol{\theta}} = \mathbf{0}$. Combining them, we obtain a set of $(n-m) \times (m + n_d \times (n-m))$ equations in $(m - n_d) \times (n-m)$ unknowns[4], $\frac{d\phi_k}{d\theta_i}$'s. Hence we can eliminate all the partial derivatives from the above *over-constrained* equations, and obtain $n_d \times (n-m) \times (n-m+1)$ *homogeneous* equations in the *architectural and configurational* parameters of the mechanism. These equations yield $n_d \times (n-m)$ conditions giving the ID criteria(see examples in the next section), and the rest N of them give an equal number of relationships between the architectural parameters involved in the *mobile* part of the mechanism, where N is given by

$$N = n_d \times (n-m)^2 \tag{21}$$

Substituting relationships obtained above in the original constraint equation(1), we obtain the configurational and architectural constraints on the dwelling part of the mechanism.

Note : The quantity N gives an upper bound on the number of architectural constraints that can be extracted using the above procedure. The actual number, however, will depend on the structure of the constraint equations, hence may be less than N (see examples in section 4).

3.3 Summary

In this section, we have presented the necessary conditions for FSM and FD. We note that in literature, the criteria for FSM has been stated as the meeting of the branches of forward kinematics as well as those of inverse kinematics[6]. We find that that the meeting of the branches of forward kinematics occurs when the matrix $[\mathbf{K}^*]$ is rank deficient, and meeting of the branches of inverse kinematics requires that one or more row(s) of $[\mathbf{K}^*]^{-1}(-[\mathbf{K}])$ is(are) null respectively. The rank deficiency of $[\mathbf{K}^*]$ and one or more row(s) of $[\mathbf{K}^*]^{-1}(-[\mathbf{K}])$ being

[4]The number of unknown derivatives is reduced by $n_d \times (n-m)$, since these many derivatives will be identically zero as n_d passive links are dwelling.

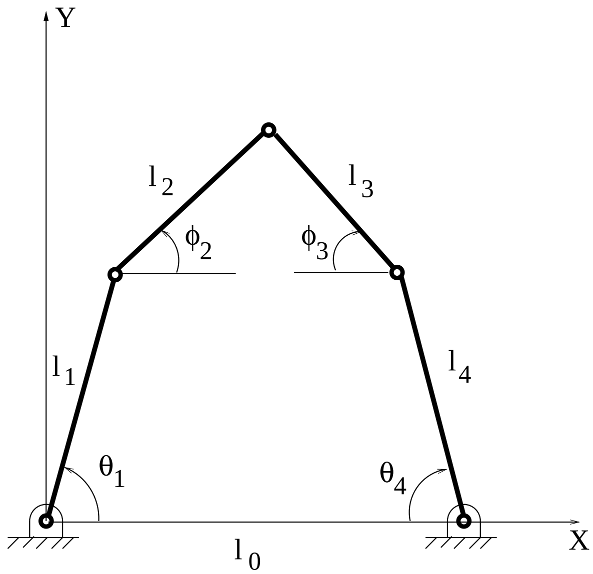

Figure 1: Planar Five-bar Mechanism with Revolute Joints

null are the criteria for gain singularity and ID respectively. The simultaneous degeneracy of $[\mathbf{K}^*]$, and $[\mathbf{K}^*]^{-1}(-[\mathbf{K}])$ is only a special case and is neither a necessary nor a sufficient condition for FSM, as we will show with an example in the following section.

4 Illustrative Examples

In this section, we illustrate the theoretical results discussed above with the help of several planar and spatial closed-loop mechanisms and parallel manipulators.

4.1 Planar Five-bar Mechanism

In this section, we study a 2-degree-of-freedom single-loop mechanism, namely the planar Five-bar mechanism with revolute joints.

Geometry of the Five-bar Mechanism

Figure 1 shows the general geometry of a Five-bar mechanism. Link 1 and link 4 are the actuated links, the *active* and *passive* variables are given by $\boldsymbol{\theta} = (\theta_1, \theta_4)^T$, $\boldsymbol{\phi} = (\phi_2, \phi_3)^T$ respectively, and $\mathbf{q} = (\theta_1, \phi_2, \phi_3, \theta_4)^T$.

Formulation of Instantaneous Kinematics

1. Loop-closure equations
   We have the loop-closure equations in the form[5]

$$\begin{aligned} \eta_1 &= l_1 c_1 + l_2 c_2 + l_3 c_3 + l_4 c_4 - l_0 = 0 \\ \eta_2 &= l_1 s_1 + l_2 s_2 - l_3 s_3 - l_4 s_4 = 0 \end{aligned} \tag{22}$$

   The above equations may be solved in closed form, and we can find $\boldsymbol{\phi}$ for any *valid* input $\boldsymbol{\theta}$.

2. Formulation of the matrices
   We compute the matrices $[\mathbf{K}]$ and $[\mathbf{K}^*]$ by differentiating the above equation with respect to appropriate variables.

$$\begin{aligned} [\mathbf{K}] &= \begin{pmatrix} -l_1 s_1 & l_4 s_4 \\ l_1 c_1 & -l_4 c_4 \end{pmatrix} \\ [\mathbf{K}^*] &= \begin{pmatrix} -l_2 s_2 & -l_3 s_3 \\ l_2 c_2 & -l_3 c_3 \end{pmatrix} \end{aligned} \tag{23}$$

Condition for Gain Singularity

From equation (9), the condition for *gain* singularity is given by

$$\begin{aligned} l_2 l_3 \sin(\phi_2 + \phi_3) &= 0 \\ \Rightarrow (\phi_2 + \phi_3) &= 0, \pi \end{aligned} \tag{24}$$

A typical singular configuration is shown in figure 2.

Gained Passive Joint Motion

In this case, $\mathcal{N}([\mathbf{K}^*]) = 1$, and the null-space of $[\mathbf{K}^*]$ is spanned by the *non-null* vector $\dot{\boldsymbol{\phi}}$, where the components of $\dot{\boldsymbol{\phi}}$ satisfy the following equations

$$\begin{aligned} \frac{\dot{\phi}_2}{l_2} + \frac{\dot{\phi}_3}{l_3} &= 0 \text{ if } \phi_2 + \phi_3 = 0 \\ \frac{\dot{\phi}_2}{l_2} - \frac{\dot{\phi}_3}{l_3} &= 0 \text{ if } \phi_2 + \phi_3 = \pi \end{aligned} \tag{25}$$

This velocity is kinematically admissible, even when the actuators are locked.

[5]We denote $\cos\theta_i$ and $\sin\theta_i$ by c_i, s_i respectively in this paper.

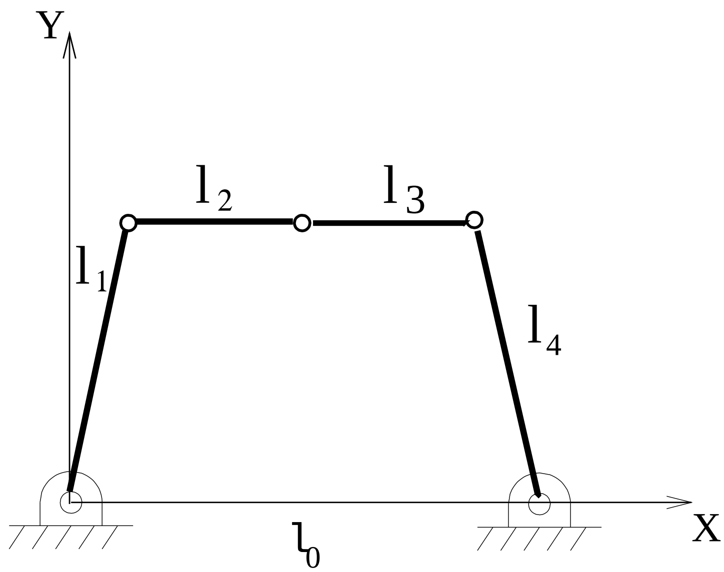

Figure 2: Singular Configuration of the Five-bar Mechanism

Locking Configuration

If $\phi_2 + \phi_3 = 0(2\pi)$, the five-bar mechanism reduces to a 4-bar mechanism instantaneously, where the coupler link is of length $l_2 + l_3$ (see figure 2). Only one of the actuators can be moved independently in this configuration. In the figure 2, it is clearly seen that the input links can not move in opposite directions.

Note : We have a certain subset of above singular configurations, in which the locking is only *instantaneous*, i.e., the mechanism can *move through* the singular configuration. One such example is $\mathbf{q} = (0,0,0,0)^T$. In this case the simultaneous degeneracy of the matrices $[\mathbf{K}]$ and $[\mathbf{K}^*]$ appears to be a *necessary criterion.*

Finite Self Motion of Five-bar Mechanism

We now investigate the gained degree-of-freedom for possible FSM. In this case, $\mathcal{N}([\mathbf{K}^*]) = 1$, hence there is only one *independent* passive variable, which we choose[6] as ϕ_2. In accordance with the notation introduced in section 3, we have $\phi^I = \phi_2, \phi^D = \phi_3$. Substituting $\boldsymbol{\eta} = (\eta_1, \eta_2)^T$ from equation (22) into (15), we get the following equations:

$$K_{11}^* + K_{12}^* \frac{d\phi_3}{d\phi_2} \quad = \quad 0$$

[6]Note that there is no *a priori* justification for choosing ϕ^I as ϕ_2. It may done based on intuition, or purely as a trial.

$$K_{21}^* + K_{22}^* \frac{d\phi_3}{d\phi_2} = 0 \tag{26}$$

where K_{ij}^* denotes the element (i, j) of the matrix $[\mathbf{K}^*]$. Consistency of this set of equations require that $K_{11}^* K_{22}^* - K_{21}^* K_{12}^* = 0$, i.e., $\det([\mathbf{K}^*]) = 0$, which is the singularity criteria. Solving the above set of equations, we get

$$\begin{aligned} \frac{d\phi_3}{d\phi_2} &= -K_{11}^*/K_{12}^* \\ \frac{d\phi_3}{d\phi_2} &= -K_{21}^*/K_{22}^* \end{aligned} \tag{27}$$

Differentiating the singularity criteria with respect to ϕ_2, we get

$$1 + \frac{d\phi_3}{d\phi_2} = 0 \tag{28}$$

Substituting for $\frac{d\phi_3}{d\phi_2}$ from equation (28) into equation (27), we get $K_{11}^* = K_{12}^*$, $K_{21}^* = K_{22}^*$, and hence finally

$$\begin{aligned} -l_2 s_2 &= -l_3 s_3 \\ l_2 c_2 &= -l_3 c_3 \end{aligned} \tag{29}$$

From the above set of equations, we can eliminate ϕ_2, ϕ_3 to obtain the *architectural requirement* on the passive part for FSM as $l_2 = l_3$, and $\phi_2 = \pi - \phi_3$. Substituting these results into the original constraint equations (22), we get

$$\begin{aligned} l_1 s_1 - l_4 s_4 &= 0 \\ l_1 c_1 + l_4 c_4 - l_0 &= 0 \end{aligned} \tag{30}$$

Elimination of θ_1 yields, after simplification and rearrangement,

$$c_4 = \frac{l_0^2 + l_4^2 - l_1^2}{2 l_0 l_4} \tag{31}$$

The last equation implies that the links $1, 0$ and 4 constitute a triangle, of which θ_4 is the angle contained by the links 0 and 4. The corresponding configuration is shown in figure 3.

We note that the FSM of the five-bar mechanism does not require the coincidence of the meetings of the branches of forward and inverse kinematics, the criterion of which are $\sin(\phi_2+\phi_3) = 0$ and $\sin(\theta_i - \phi_i) = 0$, $i = 1, 2$ respectively. In figure 3, we see that the forward kinematics branches meet at all $\phi_2 = \pi - \phi_3$, where as the inverse kinematics branches can

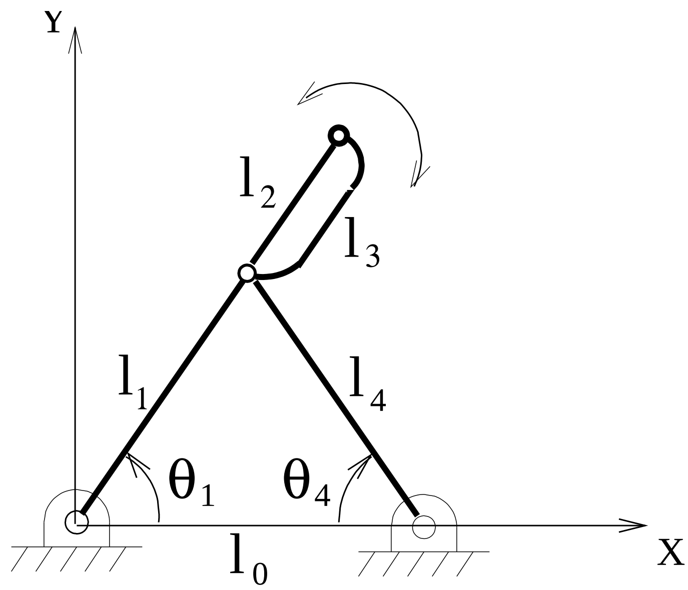

Figure 3: Configuration of the Five-bar Mechanism for FSM

meet only at two points, where that passive links come in line with either of the active links, which implies the conditions presented in [6] are not necessary. In figure 4, we show a configuration of the mechanism, where both forward and inverse kinematics branches meet, but we have $l_2 \neq l_3$, so that there is no configuration which can show FSM. This shows the conditions in [6] are not sufficient.

4.2 Planar Three-degree-of-freedom Parallel Manipulator with Revolute Joints

Geometry of the Planar three-degree-of-freedom Parallel Manipulator

Figure 5 shows the geometry of the planar three-degree-of-freedom manipulator. This three-loop mechanism has been studied in [6], in which the architecture was assumed to be symmetrical. We, however, allow for *general* linkage parameters, i.e., the individual fingers can have corresponding links of different lengths. The gripped object is assumed to be an equilateral triangle of side a. All the active links, l_i's are connected to the respective pivots, where the motors are located. The active variable in this case are given by $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, and the passive variable by $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3, \alpha)^T$, where θ_i, ϕ_i's have been shown in the figure 5. Note that the orientation of the platform, denoted by α, has also been included in $\boldsymbol{\phi}$, and we require 4 independent equations to solve for these 4 quantities.

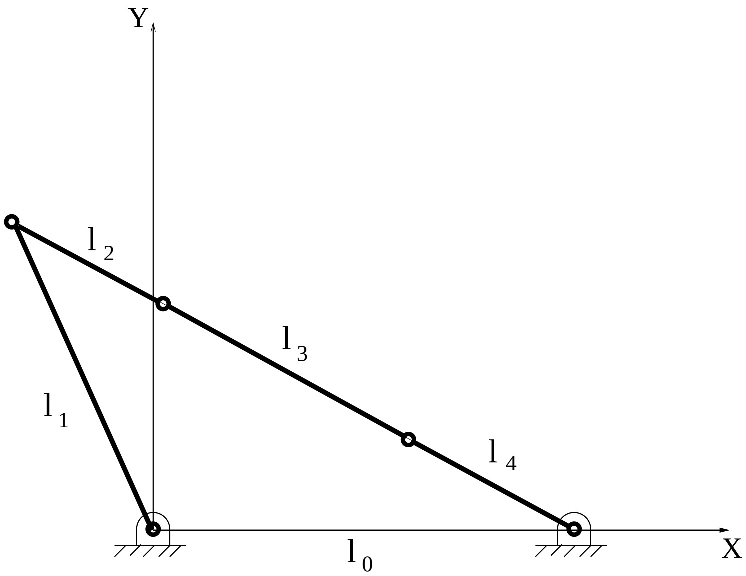

Figure 4: Meeting of the Kinematics Branches in a Five-bar Mechanism

Formulation of Instantaneous Kinematics

1. Loop-closure equations

   Loop closure equations are formed by considering the vector-loops between two pairs of fixed pivots (in particular, the pairs $(1,2)$, and $(2,3)$)[7] and are given by

$$\begin{aligned}
\eta_1 &= l_1 c_1 + r_1 c_{\phi_1} + a c_\alpha - r_2 c_{\phi_2} - l_2 c_2 - x_2 = 0 \\
\eta_2 &= l_1 s_1 + r_1 s_{\phi_1} + a s_\alpha - r_2 s_{\phi_2} - l_2 s_2 = 0 \\
\eta_3 &= x_2 + l_2 c_2 + r_2 c_{\phi_2} + a c_{(\frac{2\pi}{3}+\alpha)} - r_3 c_{\phi_3} - l_3 c_3 - x_3 = 0 \\
\eta_4 &= l_2 s_2 + r_2 s_{\phi_2} + a s_{(\frac{2\pi}{3}+\alpha)} - r_3 s_{\phi_3} - l_3 s_3 - y_3 = 0 \qquad (32)
\end{aligned}$$

   where (x_i, y_i) give the coordinates of the i^{th} pivot.

2. Formulation of the matrices

   The matrices $[\mathbf{K}]$ and $[\mathbf{K}^*]$ are obtained from the differentiation of the above equations with respect to $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ respectively, and have the following expressions.

$$[\mathbf{K}] = \begin{pmatrix} -l_1 s_1 & l_2 s_2 & 0 \\ l_1 c_1 & -l_2 c_2 & 0 \\ 0 & -l_2 s_2 & l_3 s_3 \\ 0 & l_2 c_2 & -l_3 c_3 \end{pmatrix}$$

[7]Note that we can construct a third vector equation by considering the pivot pair $(1,3)$, but it may be shown to be linearly dependent on the two previously formed equations.

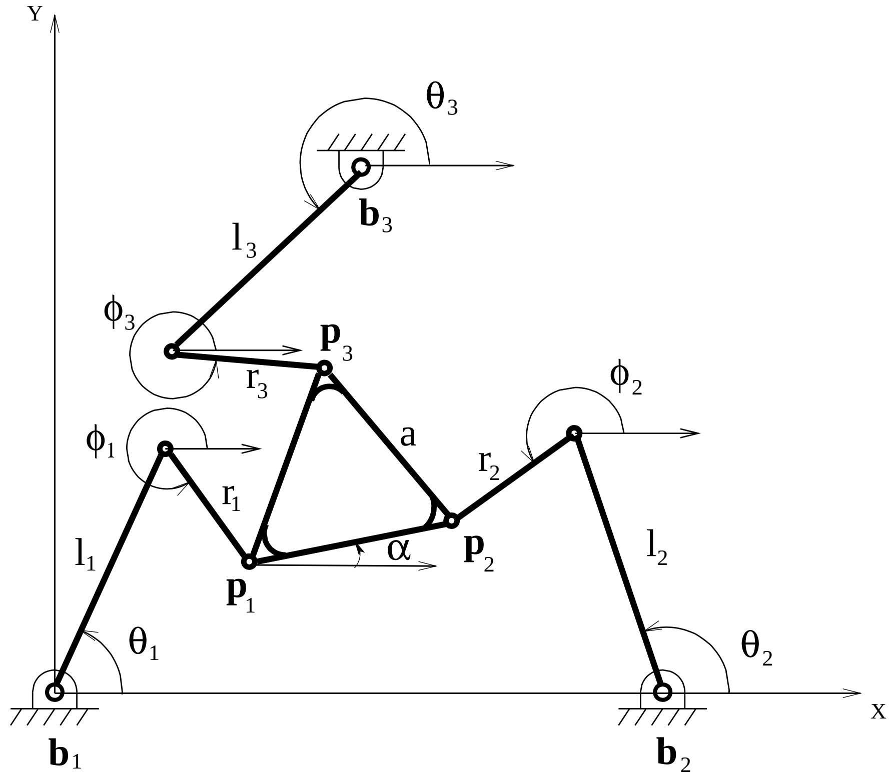

Figure 5: Planar Three-degree-of-freedom Parallel Manipulator

$$
[\mathbf{K}^*] = \begin{pmatrix} -r_1 s_{\phi_1} & r_2 s_{\phi_2} & 0 & -a s_\alpha \\ r_1 c_{\phi_1} & -r_2 c_{\phi_2} & 0 & a c_\alpha \\ 0 & -r_2 s_{\phi_2} & r_3 s_{\phi_3} & -a s_{(\frac{2\pi}{3}+\alpha)} \\ 0 & r_2 c_{\phi_2} & -r_3 c_{\phi_3} & a c_{(\frac{2\pi}{3}+\alpha)} \end{pmatrix} \tag{33}
$$

Condition for Gain Singularity

From equation (9), the condition for gain-singularity is given by

$$
a r_1 r_2 r_3 (\sin(\phi_1 - \alpha)\sin(\phi_2 - \phi_3) + \sin(\phi_1 - \phi_2)\sin(\alpha + \frac{2\pi}{3} - \phi_3)) = 0 \tag{34}
$$

which shows that the singular configurations lie on a 1-dimensional sub-manifold of $\mathcal{C}$. We identify two special classes of singular configurations from the above expression:

1. All passive links are parallel.

2. All the passive links, or their hypothetical extensions, intersect at a point.

Gained Passive Joint Motion

In both the above cases, $\mathcal{N}([\mathbf{K}^*]) = 1$, and the null-space of $[\mathbf{K}^*]$ is spanned by a single *non-null* vector $\dot{\boldsymbol{\phi}}$, which is different in the two cases.

1. If all passive links are parallel, then the null-space is spanned by $(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, 0)^T$.

2. If the passive links intersect at a point, the components of $\dot{\boldsymbol{\phi}}$ depend on the configuration. In particular, when all the passive links, or their hypothetical extensions intersect at the center of the mobile platform, the gained passive velocity is given by $(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, -\frac{\sqrt{3}}{a})^T$

Note that for the first case, the angular velocity of the platform, given by $\dot{\alpha}$, is zero. However, in the second case, the platform is allowed to have an angular velocity instantaneously at a singular point. We now investigate the condition on the architecture of the mechanism for this angular velocity to result in finite rotation.

Finite Self Motion

As noted above, we have $\mathcal{N}([\mathbf{K}^*]) = 1$, and the only *independent* passive variable is chosen to be α. Hence we have $\phi^I = \alpha$, $\phi^D = (\phi_1, \phi_2, \phi_3)^T$. Equating the total derivative of the constraint equations with respect to α to zero, we get the following equations

$$\begin{aligned}
K^*_{11}\frac{d\phi_1}{d\alpha} + K^*_{12}\frac{d\phi_2}{d\alpha} + K^*_{14} &= 0 \\
K^*_{21}\frac{d\phi_1}{d\alpha} + K^*_{22}\frac{d\phi_2}{d\alpha} + K^*_{24} &= 0 \\
K^*_{32}\frac{d\phi_2}{d\alpha} + K^*_{33}\frac{d\phi_3}{d\alpha} + K^*_{34} &= 0 \\
K^*_{42}\frac{d\phi_2}{d\alpha} + K^*_{43}\frac{d\phi_3}{d\alpha} + K^*_{44} &= 0
\end{aligned} \tag{35}$$

Solving the first two equations simultaneously, we get

$$\begin{aligned}
\frac{d\phi_1}{d\alpha} &= \frac{a/\sin(\phi_1 - \phi_2)}{r_1/\sin(\phi_2 - \alpha)} \\
\frac{d\phi_2}{d\alpha} &= \frac{a/\sin(\phi_1 - \phi_2)}{r_2/\sin(\phi_1 - \alpha)}
\end{aligned} \tag{36}$$

Similarly, solving the last two equations simultaneously, we get

$$\begin{aligned}
\frac{d\phi_2}{d\alpha} &= \frac{a/\sin(\phi_2 - \phi_3)}{r_2/\sin(\phi_3 - \alpha - \frac{2\pi}{3})} \\
\frac{d\phi_3}{d\alpha} &= \frac{a/\sin(\phi_2 - \phi_3)}{r_3/\sin(\phi_2 - \alpha - \frac{2\pi}{3})}
\end{aligned} \tag{37}$$

It may be verified that equating the two expressions of $\frac{d\phi_2}{d\alpha}$, we can recover the singularity criterion given by equation (34), i.e., the consistency of these equations requires $\det[\mathbf{K}]^* = 0$. Differentiating equation (34) with respect to α, we get

$$\begin{aligned}
&\cos(\phi_1-\alpha)(\frac{d\phi_1}{d\alpha}-1)\sin(\phi_2-\phi_3)+\sin(\phi_1-\alpha)\cos(\phi_2-\phi_3)(\frac{d\phi_2}{d\alpha}-\frac{d\phi_3}{d\alpha}) \\
&+\cos(\phi_1-\phi_2)(\frac{d\phi_1}{d\alpha}-\frac{d\phi_2}{d\alpha})\sin(\frac{2\pi}{3}+\alpha-\phi_3) \\
&+\sin(\phi_1-\phi_2)\cos(\frac{2\pi}{3}+\alpha-\phi_3)(1-\frac{d\phi_3}{d\alpha})=0
\end{aligned} \tag{38}$$

It may be seen that the last equation is satisfied by

$$\frac{d\phi_i}{d\alpha}=1,\ i=1,2,3 \tag{39}$$

and since equations(35, 38) are linear in $\frac{d\phi_i}{d\alpha}$, $\frac{d\phi_i}{d\alpha}=1$ is the *only* solution of the system. We now interpret the solution from a geometric viewpoint. From equation(36), we get

$$\frac{a}{\sin(\phi_1-\phi_2)}=\frac{r_1}{\sin(\phi_2-\alpha)}=\frac{r_2}{\sin(\phi_1-\alpha)} \tag{40}$$

The last equation indicates that the links of length r_1, r_2 and a side of the mobile platform form a triangle, where the angles opposite to the sides r_1, r_2 and a are related to the angles $(\phi_2-\alpha), (\phi_1-\alpha)$ and $(\phi_1-\phi_2)$ respectively (see figure 6). Similarly, from equation(37), we find that the links r_2, r_3 form another triangle with another side of the mobile platform. Geometrically, the above conditions require that the ends of the passive links r_1, r_2, r_3 meet at a point, and in this configuration, the passive links can rotate finitely with the platform about this point at the same rate, even when the actuators are held fixed. This observation verifies equation(39).

We observe that these requirements are more general than those presented in [6], and for FSM, the meeting of the branches of inverse kinematics is *not* required. In fact, meeting of the inverse kinematics branches lead to ID, as we will show in our discussion on ID.

We now find out the configuration of the active links for FSM. Using equation(39) in equation (35), and substituting in the original constraint equation(32), we get

$$\begin{aligned}
l_1c_1-l_2c_2-x_2 &= 0\\
l_1s_1-l_2s_2 &= 0\\
x_2+l_2c_2-l_3c_3-x_3 &= 0\\
l_2s_2-l_3s_3-y_3 &= 0
\end{aligned} \tag{41}$$

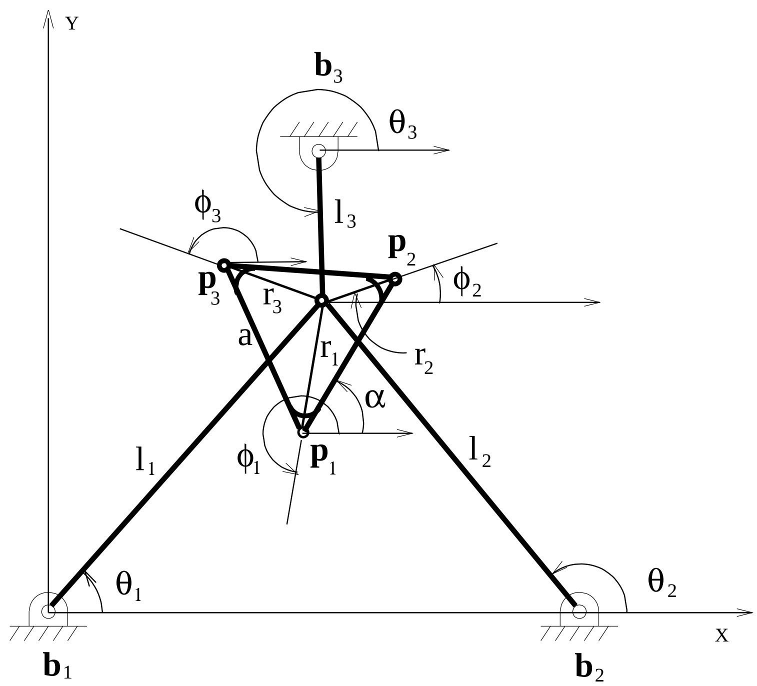

Figure 6: Configuration of the Three-degree-of-freedom Parallel Manipulator for FSM

Eliminating θ_1 from the first two of the above equations, we get after some rearrangement,

$$\cos(\pi - \theta_2) = \frac{l_2^2 + x_2^2 - l_1^2}{2 l_2 x_2} \tag{42}$$

which shows that the links l_1, l_2 form a triangle with a side of the base triangle, of which, $\pi - \theta_2$ is the angle opposite to the link l_1. Similarly, from the last two equations of (41), we find that the links l_2, l_3 form a triangle with another side of the base triangle. Combining these two conditions, we find that the tips of the active links also meet at a point, which is also the point of meeting of the ends of the passive links, hence the results are consistent. Note that in this case, the architectural requirement is not unique, and any set of link lengths that allow the mechanism to attain the special configuration is sufficient to allow FSM. The configuration of the manipulator corresponding to the above conditions is shown in figure 6.

Finite Dwell

The mobile platform of the three-degree-of-freedom parallel manipulator can also show finite dwell. The architectural requirements for this is derived below using the method described in section 3.

The row of the matrix $[\mathbf{K}^*]^{-1}(-[\mathbf{K}])$ corresponding to $\dot{\alpha}$ is given by

$$\frac{r_1 r_2 r_3}{\det[\mathbf{K}^*]} \begin{pmatrix} l_1 \sin(\phi_1 - \theta_1) \sin(\phi_2 - \phi_3) \\ l_2 \sin(\phi_2 - \theta_2) \sin(\phi_3 - \phi_1) \\ l_3 \sin(\phi_3 - \theta_3) \sin(\phi_1 - \phi_2) \end{pmatrix}^T$$

where $\det[\mathbf{K}^*]$ is given in the left-hand side of equation (34). Since we assume $\det[\mathbf{K}^*] \neq 0$ for the above to be defined, we must have the ϕ_i's all different, i.e., we have

$$\sin(\phi_i - \theta_i) = 0 \quad i = 1, 2, 3 \tag{43}$$

The above equation gives the ID criteria, which implies that the individual fingers are stretched (or folded back) in this configuration. This is also the requirement of meeting of the inverse kinematics branches. Differentiating equation(43) with respect to the active variables θ_j's, and solving for the partial derivatives $\frac{\partial \phi_i}{\partial \theta_j}$, we get

$$\frac{\partial \phi_i}{\partial \theta_j} = \delta_{ij} \tag{44}$$

Differentiating the constraint equations with respect to θ_1, and noting the results in equation(44), we obtain

$$\begin{aligned} -l_1 s_1 - r_1 s_{\phi_1} &= 0 \\ l_1 c_1 + r_1 c_{\phi_1} &= 0 \end{aligned} \tag{45}$$

where from we recover the architectural requirement, $l_1 = r_1$, and the ID criterion for this finger as $\phi_1 = \pi + \theta_1$. Proceeding in a similar fashion, we obtain the architectural constraint on the links of the i^{th} finger as $l_i = r_i$. Note that there is no relationship between the link-lengths of different fingers. Also note that maximum number of architectural requirements on the *non-dwelling* part of the manipulator, as predicted by equation (21) is $1 \times (7-4)^2 = 9$. In this case, however, the actual number is only 3.

To find the relation between the size of the mobile platform a, and the base platform formed by the pivots, we substitute $l_i = r_i$, and $\phi_i = \pi + \theta_i$ in the constraint equation(32) and obtain

$$\begin{aligned} a c_\alpha - x_2 &= 0 \\ a s_\alpha &= 0 \\ x_2 + a c_{(\frac{2\pi}{3}+\alpha)} - x_3 &= 0 \\ a s_{(\frac{2\pi}{3}+\alpha)} - y_3 &= 0 \end{aligned} \tag{46}$$

From the last equation, we find that $\alpha = 0, x_2 = a$ and $(x_3 - x_2)^2 + y_3^2 = a^2$. Geometrically, this shows that the mobile platform has the same size as the base platform, and rests on top of the base for finite dwell. All the passive links fold back on the corresponding active links, such that their tips coincide with the pivots.

4.3 Spatial Three-Fingered Manipulator

We now analyze a three-loop six-degree-of-freedom spatial manipulator.

Geometry of the Spatial Three-Fingered Manipulator

The manipulator has three-fingers, each of which has three revolute joints. The first two of the joints in each finger are actuated, and the last link is passive. The active variable is given by $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \psi_1, \psi_2, \psi_3)^T$, and the passive variable by $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^T$. The *DH parameters* of the i^{th} finger is given in the following table.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_i
2	$\frac{\pi}{2}$	l_{i1}	0	ψ_i
3	0	l_{i2}	0	ϕ_i

The base of the third finger is taken to be the origin of the fixed coordinate system, and the base of other two fingers are placed symmetrically about the X axis at distance d from it, and at a height h above the origin, i.e., the base coordinates of the three fingers are given by $\mathbf{b}_1 = (0, -d, h)^T, \mathbf{b}_2 = (0, d, h)^T$, and $\mathbf{b}_3 = (0, 0, 0)^T$ respectively. Finally, the gripped object is modeled as an equilateral triangular platform of side s, and the connection of the object with the fingers are modeled as spherical joints.

Formulation of Instantaneous Kinematics

1. Loop-Closure Equations

   The position vectors $\mathbf{p}_i$ $(i = 1, 2, 3)$ of the centers of the spherical joints are given by

   $$\mathbf{p}_i = \mathbf{b}_i + \mathbf{x}_i \tag{47}$$

   where $\mathbf{x}_i$ denotes the position of the center of the spherical joint attached to the i^{th} finger with respect to the base of the finger. We have

   $$\mathbf{x}_i = \begin{pmatrix} c_{\theta_i}(l_{i1} + l_{i2}c_{\psi_i} + l_{i3}c_{(\psi_i+\phi_i)}) \\ s_{\theta_i}(l_{i1} + l_{i2}c_{\psi_i} + l_{i3}c_{(\psi_i+\phi_i)}) \\ l_{i2}s_{\psi_i} + l_{i3}s_{(\psi_i+\phi_i)} \end{pmatrix}$$

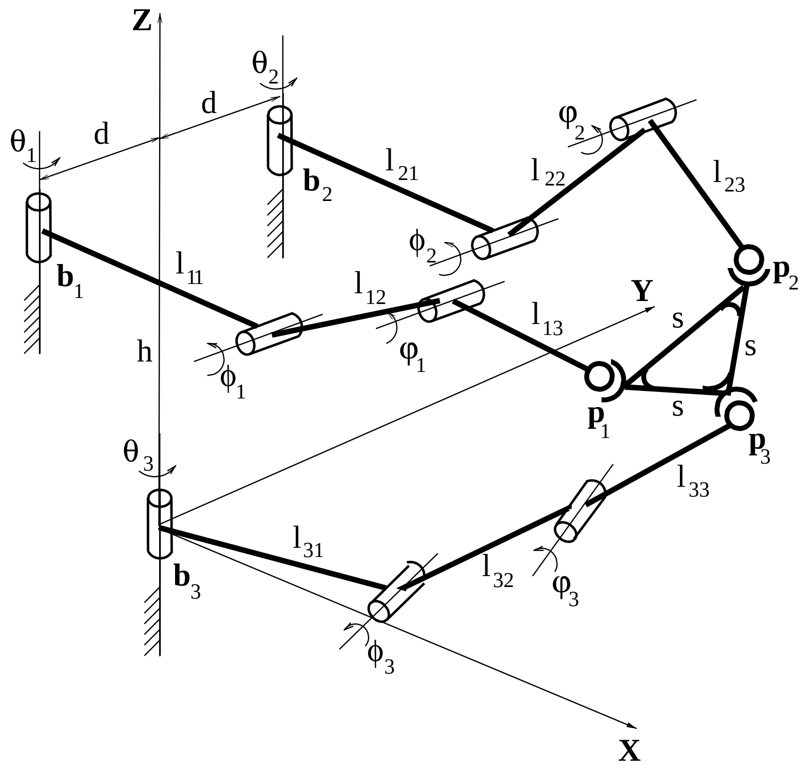

Figure 7: Spatial Three-Fingered Manipulator

The loop-closure equations are formed by equating the distance between the tips of pairs of fingers with the length of a side of the gripped object.

$$\begin{aligned} (\mathbf{p}_2 - \mathbf{p}_1) \cdot (\mathbf{p}_2 - \mathbf{p}_1) &= s^2 \\ (\mathbf{p}_3 - \mathbf{p}_1) \cdot (\mathbf{p}_3 - \mathbf{p}_1) &= s^2 \\ (\mathbf{p}_2 - \mathbf{p}_3) \cdot (\mathbf{p}_2 - \mathbf{p}_3) &= s^2 \end{aligned} \tag{48}$$

2. Formulation of the Matrices

The matrix $[\mathbf{K}^*]$ has the form:

$$[\mathbf{K}^*] = \begin{pmatrix} K^*_{11} & K^*_{12} & 0 \\ 0 & K^*_{22} & K^*_{23} \\ K^*_{31} & 0 & K^*_{33} \end{pmatrix}$$

where the non-zero entries of $[\mathbf{K}^*]$ are given by

$$K^*_{11} = 2(\mathbf{b}_{12} + \mathbf{x}_{12}) \cdot \frac{\partial \mathbf{x}_1}{\partial \phi_1}$$

$$\begin{aligned} K^*_{12} &= -2(\mathbf{b}_{12}+\mathbf{x}_{12})\cdot\frac{\partial\mathbf{x}_2}{\partial\phi_2} \\ K^*_{22} &= 2(\mathbf{b}_{23}+\mathbf{x}_{23})\cdot\frac{\partial\mathbf{x}_2}{\partial\phi_2} \\ K^*_{23} &= -2(\mathbf{b}_{23}+\mathbf{x}_{23})\cdot\frac{\partial\mathbf{x}_3}{\partial\phi_3} \\ K^*_{31} &= -2(\mathbf{b}_{31}+\mathbf{x}_{31})\cdot\frac{\partial\mathbf{x}_1}{\partial\phi_1} \\ K^*_{33} &= 2(\mathbf{b}_{31}+\mathbf{x}_{31})\cdot\frac{\partial\mathbf{x}_3}{\partial\phi_3} \end{aligned} \tag{49}$$

and $\mathbf{b}_{ij}$ and $\mathbf{x}_{ij}$ denote $\mathbf{b}_i - \mathbf{b}_j$ and $\mathbf{x}_i - \mathbf{x}_j$ respectively.

Condition for Gain Singularity

The singularity criteria is given by

$$\det[\mathbf{K}^*] = K^*_{11}K^*_{22}K^*_{33} - K^*_{12}K^*_{23}K^*_{31} = 0 \tag{50}$$

Finite Self Motion of the Spatial Six-degree-of-freedom Mechanism

We study the case when $\mathcal{N}([\mathbf{K}^*]) = 1$ and choose $\phi^I = \phi_1$, $\phi^D = (\phi_2, \phi_3)^T$. Equating the total derivatives of the constraint equations with respect to ϕ_1 to zero, we get

$$\begin{aligned} K^*_{11} + K^*_{12}\frac{d\phi_2}{d\phi_1} &= 0 \\ K^*_{22}\frac{d\phi_2}{d\phi_1} + K^*_{23}\frac{d\phi_3}{d\phi_1} &= 0 \\ K^*_{31}\frac{d\phi_3}{d\phi_1} + K^*_{33} &= 0 \end{aligned} \tag{51}$$

Solving the first and the last equation respectively, we get

$$\begin{aligned} \frac{d\phi_2}{d\phi_1} &= -K^*_{11}/K^*_{12} \\ \frac{d\phi_3}{d\phi_1} &= -K^*_{31}/K^*_{33} \end{aligned} \tag{52}$$

Substituting these solutions into the second equation of (51) we recover the singularity criteria, i.e., equation (52) is consistent at a gain-singularity. Differentiating equation (50) with respect to ϕ_1, we get an equation involving complicated expressions. However, as in the case of the planar three-degree-of-freedom manipulator, it may be verified that this equation is satisfied by

$$\frac{d\phi_i}{d\phi_1} = 1,\ i = 1, 2, 3 \tag{53}$$

These equations being linear in $\frac{d\phi_i}{d\phi_1}$, $\frac{d\phi_i}{d\phi_1} = 1$ is the *only* solution of the system. From equations (50, 51), we get

$$\begin{aligned} K^*_{11} &= -K^*_{12} \\ K^*_{22} &= -K^*_{21} \\ K^*_{33} &= -K^*_{31} \end{aligned} \tag{54}$$

Combining the last equation with equation (49), we obtain

$$(\mathbf{b}_{12} + \mathbf{x}_{12}) \cdot (\frac{\partial \mathbf{x}_1}{\partial \phi_1} - \frac{\partial \mathbf{x}_2}{\partial \phi_2}) = 0 \tag{55}$$

whence we get $\frac{\partial \mathbf{x}_1}{\partial \phi_1} = \frac{\partial \mathbf{x}_2}{\partial \phi_2}$, since in general, the vector $(\mathbf{b}_{12} + \mathbf{x}_{12})$ is not perpendicular to $(\frac{\partial \mathbf{x}_1}{\partial \phi_1} - \frac{\partial \mathbf{x}_2}{\partial \phi_2})$. Similarly, $\frac{\partial \mathbf{x}_2}{\partial \phi_2} = \frac{\partial \mathbf{x}_3}{\partial \phi_3}$. Using the expressions of $\mathbf{x}_i$, we get

$$\begin{aligned} l_{13}\cos\theta_1 \sin(\phi_1 + \psi_1) &= l_{23}\cos\theta_2 \sin(\phi_2 + \psi_2) \\ l_{13}\sin\theta_1 \sin(\phi_1 + \psi_1) &= l_{23}\sin\theta_2 \sin(\phi_2 + \psi_2) \\ l_{13}\cos(\phi_1 + \psi_1) &= l_{23}\cos(\phi_2 + \psi_2) \end{aligned} \tag{56}$$

Squaring both sides of all the above equations and adding, we get $l_{13} = l_{23}$. Using this, from the first two equations of the set, we obtain $\theta_1 = \theta_2$, and finally, $\phi_2 + \psi_2 = \phi_1 + \psi_1$. Proceeding in the same manner, from $\frac{\partial \mathbf{x}_1}{\partial \phi_1} = \frac{\partial \mathbf{x}_3}{\partial \phi_3}$ we obtain $l_{13} = l_{33}$, $\theta_1 = \theta_3$, and $\phi_1 + \psi_1 = \phi_3 + \psi_3$. This implies that the passive links of all the fingers are of the same size, and they make the same angle with the horizontal, and the fingers lie in parallel vertical planes which make an angle θ with the $\mathbf{X}$ axis, where $\theta = \theta_1 = \theta_2 = \theta_3$. The configurational requirement of FSM is non-unique in this case, and there is at least one single parameter (θ) family of solutions to the FSM equations.

Substituting the relations derived above in the first of the constraint equations (47) we get

$$(l_{11}+l_{12}c_{\psi_1}-l_{21}-l_{22}c_{\psi_2})^2 c_\theta^2+(2d+(l_{11}+l_{12}c_{\psi_1}-l_{21}-l_{22}c_{\psi_2})s_\theta)^2+(l_{12}s_{\psi_1}-l_{22}s_{\psi_2})^2 = s^2 \tag{57}$$

The solution is valid for all θ satisfying the original constraint equations, and may be non-unique in ψ_1 and ψ_2, and hence also in the linkage parameters of the first two links of the fingers of the manipulators. In particular, if we choose identical linkage parameters and symmetric configuration, i.e., $l_{11} = l_{21}$, $l_{12} = l_{22}$ and $\psi_1 = \psi_2$, it is easy to see that $s = 2d$. Similarly, from the third constraint equation, we get

$$(l_{11}+l_{12}c_{\psi_1}-l_{31}-l_{32}c_{\psi_3})^2c_\theta^2+(d+(l_{11}+l_{12}c_{\psi_1}-l_{31}-l_{32}c_{\psi_3})s_\theta)^2+(h+l_{12}s_{\psi_1}-l_{32}s_{\psi_3})^2 = s^2 \quad (58)$$

Under a similar assumption of symmetry, i.e., $l_{11} = l_{31}, l_{12} = l_{32}$ and $\psi_1 = \psi_3$, we get $h = \sqrt{3}d$.

Thus one family of FSM requires $s = 2d, h = \sqrt{3}d, \theta_1 = \theta_2 = \theta_3$ and $l_{i1} = l_{i2} = l_{i3}, \quad i = 1, 2, 3$.

4.4 Planar Six-bar Mechanism

In this section, we study a single-degree-of-freedom planar six-bar mechanism derived from the Watt chain. This is an example of a mechanism where $\mathcal{N}([\mathbf{K}^*])$ can be 1 or 2.

Geometry of the Six-bar Mechanism

Figure 8 shows the general geometry of a six-bar mechanism. Link 1 is the actuated

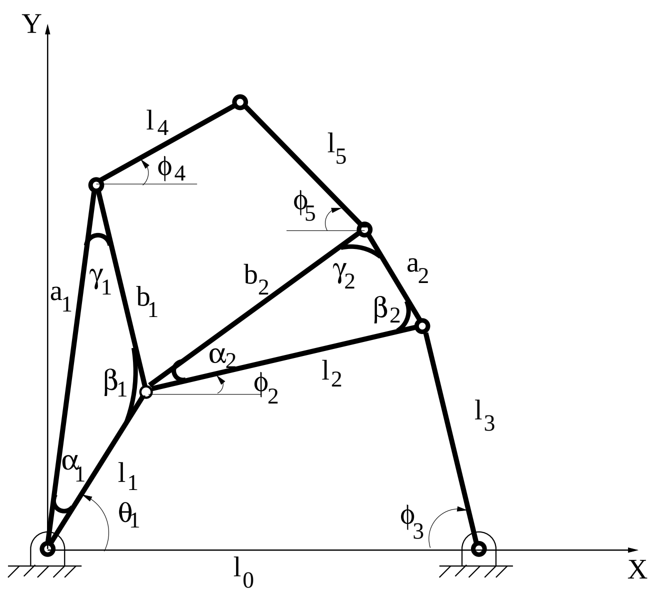

Figure 8: Planar Six-bar Mechanism with Revolute Joints

link, and the *active variable* is given by $\boldsymbol{\theta} = \theta_1$, while passive variables are given by $\boldsymbol{\phi} = (\phi_2, \phi_3, \phi_4, \phi_5)^T$.

Formulation of Instantaneous Kinematics

1. Loop-closure equations

   We have the loop-closure equations in the form

$$\begin{aligned}\eta_1 &= l_1 c_1 + l_2 c_{\phi_2} + l_3 c_{\phi_3} - l_0 = 0\\ \eta_2 &= l_1 s_1 + l_2 s_{\phi_2} - l_3 s_{\phi_3} = 0\\ \eta_3 &= -b_1\cos(\theta_1 - \beta_1) + l_4 c_{\phi_4} + l_5 c_{\phi_5} - b_2\cos(\phi_2 + \alpha_2) = 0\\ \eta_4 &= -b_1\sin(\theta_1 - \beta_1) + l_4 s_{\phi_4} - l_5 s_{\phi_5} - b_2\sin(\phi_2 + \alpha_2) = 0\end{aligned} \tag{59}$$

2. Formulation of the matrices

   We find the matrix $[\mathbf{K}^*]$ from the above equation as

$$[\mathbf{K}^*] = \begin{pmatrix} -l_2 s_{\phi_2} & -l_3 s_{\phi_3} & 0 & 0\\ l_2 c_{\phi_2} & -l_3 c_{\phi_3} & 0 & 0\\ b_2\sin(\phi_2+\alpha_2) & 0 & -l_4 s_{\phi_4} & -l_5 s_{\phi_5}\\ -b_2\cos(\phi_2+\alpha_2) & 0 & l_4 c_{\phi_4} & -l_5 c_{\phi_5}\end{pmatrix} \tag{60}$$

Condition for Gain Singularity

From equation (9), the condition for *gain* singularity is given by

$$\begin{aligned} & l_2 l_3 l_4 l_5 \sin(\phi_2+\phi_3)\sin(\phi_4+\phi_5) = 0\\ \Rightarrow\quad & (\phi_2+\phi_3) = 0,\pi \quad \text{and/or} \quad (\phi_4+\phi_5) = 0,\pi \end{aligned} \tag{61}$$

Note that $[\mathbf{K}^*]$ is of rank 2 when both $\sin(\phi_2+\phi_3) = 0$ and $\sin(\phi_4+\phi_5) = 0$.

Gained Passive Joint Motion

In this case, $\mathcal{N}([\mathbf{K}^*]) = 1$ or 2. The gained passive motion is given by $(\pm l_3 l_4 l_5 \sin(\phi_4 + \phi_5), l_2 l_4 l_5 \sin(\phi_4+\phi_5), b_2 l_3 l_5 \sin(\alpha_2 - \phi_3 - \phi_5), b_2 l_3 l_4 \sin(\alpha_2 - \phi_3 - \phi_4))^T$, when the inner loop consisting of links 1,2,3 show a gain of degree-of-freedom. The corresponding vector for the outer loop (consisting of links 1,4,5,2) is $(0, 0, \pm l_5, \mp l_4)^T$. When both the loops are at respective singular configurations, two of the eigenvalues of $[\mathbf{K}^*]$ vanish, and the corresponding eigenvectors, which span the null-space of $[\mathbf{K}^*]$ are linearly dependent. In this case, the null-space turns out to be the same as in the case of the singularity of the outer loop.

Locking Configuration

The mechanism gets locked when either $\phi_2 = -\phi_3$ or $\phi_4 = -\phi_5$. We show a special case in figure 9, where both the conditions are met.

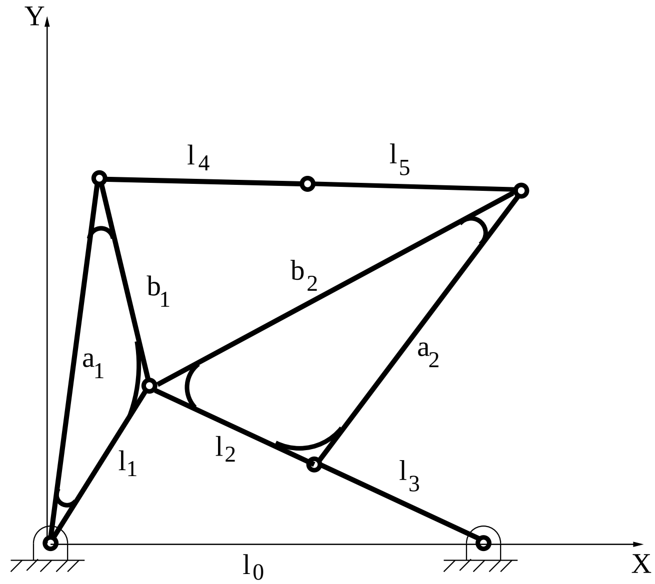

Figure 9: Configuration of the Six-bar Mechanism for Instantaneous Gain of Two Degrees of Freedom

Finite Self Motion of Six-bar Mechanism

We now investigate the gained degree-of-freedom for possible FSM. In this case, $\mathcal{N}([\mathbf{K}^*]) = 1$ or 2, hence there are several different possibilities:

1. Instantaneous gain of one degree-of-freedom
2. Instantaneous gain of two degrees-of-freedom
3. Instantaneous gain of one degree-of-freedom resulting in FSM

It was found that both the instantaneous gains can not lead to simultaneous FSM in this mechanism, hence we consider the case (3). First, we assume $\sin(\phi_2 + \phi_3) \neq 0$, and $\phi_4 + \phi_5 = \pi$. Proceeding as in section 3, with $\phi^I = \phi_4$, we get

$$\begin{aligned}
K^*_{11}\frac{d\phi_1}{d\phi_4} + K^*_{12}\frac{d\phi_3}{d\phi_4} &= 0 \\
K^*_{21}\frac{d\phi_2}{d\phi_4} + K^*_{22}\frac{d\phi_3}{d\phi_4} &= 0 \\
K^*_{31}\frac{d\phi_2}{d\phi_4} + K^*_{33} + K^*_{34}\frac{d\phi_5}{d\phi_4} &= 0 \\
K^*_{41}\frac{d\phi_2}{d\phi_4} + K^*_{43} + K^*_{44}\frac{d\phi_5}{d\phi_4} &= 0
\end{aligned} \tag{62}$$

Consistency of this set of equations requires $\det([\mathbf{K}^*]) = 0$. Following our assumption,

$$\begin{aligned} \frac{d\phi_2}{d\phi_4} &= 0 \\ \frac{d\phi_3}{d\phi_4} &= 0 \\ \frac{d\phi_5}{d\phi_4} &= -K^*_{33}/K^*_{34} \\ \frac{d\phi_5}{d\phi_4} &= -K^*_{43}/K^*_{44} \end{aligned}$$

Derivative of the singularity criteria gives $\frac{d\phi_5}{d\phi_4} = -1$. The last two equations yield $l_4 = l_5$ after some manipulation. Substituting this result in the original constraint equation (59), we get

$$\begin{aligned} -b_1 \cos(\theta_1 - \beta_1) &= b_2 \cos(\alpha_2 + \phi_2) \\ -b_1 \sin(\theta_1 - \beta_1) &= b_2 \sin(\alpha_2 + \phi_2) \end{aligned} \tag{63}$$

These equations yield $b_1 = b_2$, whence $\theta_1 - \beta_1 = \pi + \phi_2 + \alpha_2$. The corresponding configuration is shown in figure 10.

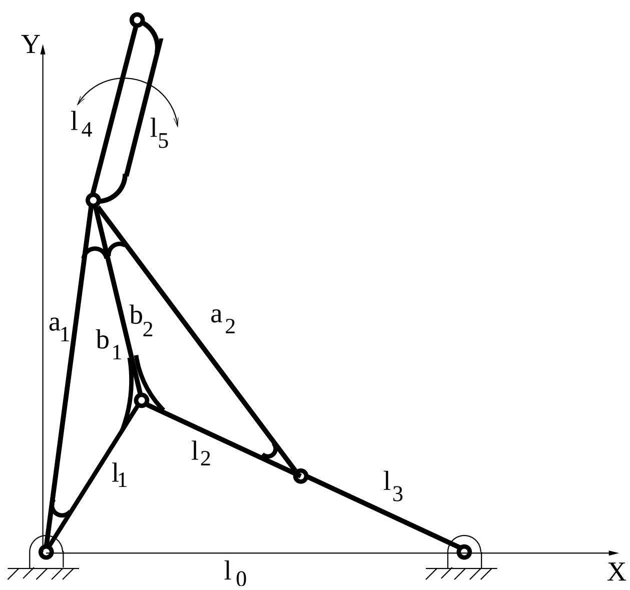

Figure 10: Configuration of the Six-bar Mechanism for FSM of the Outer Chain

We now consider the case $\sin(\phi_2 + \phi_3) = 0$, with $\phi^I = \phi_2$. Proceeding as above, we obtain $l_2 = l_3$ as the architectural requirement. Using this result, from the original constraints equations we recover $l_1 = l_0$ and $\theta_1 = 0$. The requirement on other links is such that link 2 can behave like a crank, and links 4, 5 complete a 4-bar with link 1 as the fixed link. Figure 11 shows the corresponding configuration, with link 1 lying on top of the base link, and link 3 lying on the side of length l_2 link 2. Links 2 and 3 can now rotate together finitely, while the driving link (link 1) is held fixed.

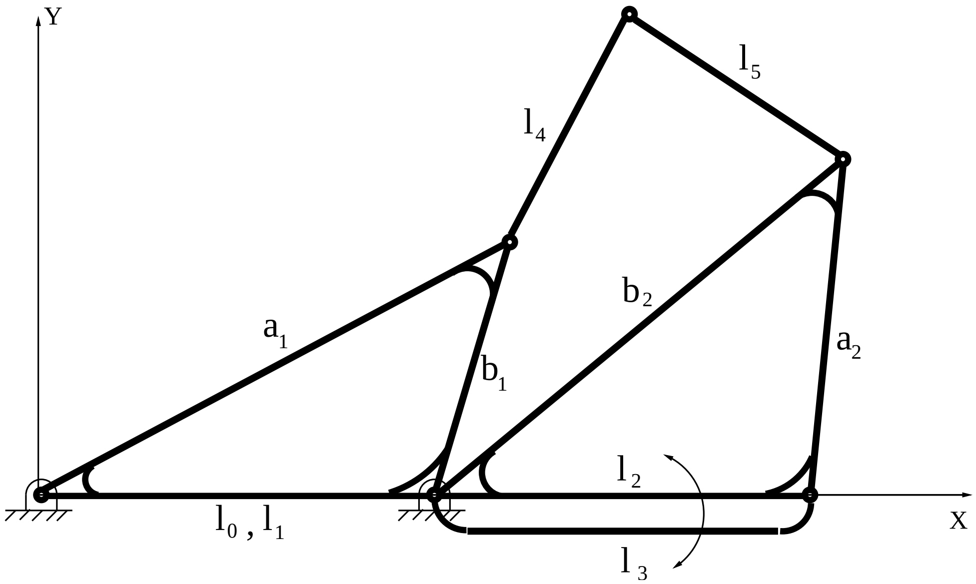

Figure 11: Configuration of the Six-bar Mechanism for FSM of the Inner Chain

5 Conclusion

In this paper, we have analyzed the singularities associated with the configuration and the architecture of parallel manipulators. The *gain* of degree-of-freedom of a closed-loop mechanism at a singularity has been analyzed both from geometric and algebraic points of view, and it has been shown to be associated with the locking of the actuators. The special singular phenomena such as finite self motion and finite dwell of a mechanism have been explained in terms of the second-order properties of the constraints, and the analytical criteria for the same have been derived. The theoretical results have been illustrated through several planar and spatial single- and multi-degree-of-freedom mechanisms and parallel manipulators.

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