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KINEMATIC ANALYSIS OF THREE-LINK SPATIAL MECHANISMS CONTAINING SPHERE-PLANE AND SPHERE-GROOVE PAIRS

GEORGE N. SANDOR†
220 MEB, University of Florida, Gainesville, FL 32611, U.S.A.

DILIP KOHLI‡
University of Wisconsin, Milwaukee, WI 53202, U.S.A.

and

MANUEL HERNANDEZ, Jr.§ and ASHITAVA GHOSAL§
University of Florida, Gainesville, FL 32611, U.S.A.

Abstract—Kinematic pairs in a spatial mechanism are viewed either as allowing relative screw motion between links or as constraining the motion of the two chains of the mechanism connected to the two elements of the pair. Using pair geometry constraints of the sphere-plane and sphere-groove kinematic pairs, the displacement, velocity and acceleration equations are derived for, *R–Sp–R*,¶ *R–Sp–P*, *P–Sp–P*, *P–Sp–R* and *R–Sg–C* three-link mechanisms. For known values of the input variable, other variables are computed in closed form. The analysis procedures are illustrated using numerical examples.

1. INTRODUCTION

The mechanisms containing higher pairs such as cams, sphere–plane, sphere–groove, or cylinder–plane provide the designer with the capabilities of designing machines and mechanisms to satisfy more complex and exact functional requirements than feasible with only lower pair mechanisms. These mechanisms in general are compact and contain fewer links than those with lower pairs.

In recent years, there has been considerable development in the tools for kinematic analysis of spatial mechanisms containing lower pairs.

Kinematic analysis of space mechanisms was initiated by the significant contribution of Dimentberg[1]. Dimentberg[2, 3] demonstrated the use of dual numbers and screw calculus to obtain closed-form displacement relationships of an *RCCC*¶ and other four-, five-, six- and seven-link spatial mechanisms containing revolute, cylinder, prismatic and helical pairs. Denavit[4] derived closed-form displacement relationships for a spatial *RCCC* mechanism using dual Euler angles. Yang[5] also derived such relationships for *RCCC* mechanisms using dual quaternions.

Vectors were first used by Chace[6] to derive closed-form displacement relations of *RCCC* mechanisms. Wallace and Freudenstein[7] also used vectors to obtain closed-form displacement relations of *RRSRR* and RRP_LRR mechanisms.

Yang[8] proposed a general formulation using dual numbers to conduct displacement analysis of *RCRCR* spatial five-link mechanisms. Soni and Pamidi[9] extended this application of (3 × 3) matrices with dual elements to obtain closed-form displacement relations of *RCCRR* mechanisms.

Yuan[10] employed screw coordinates to obtain closed-form displacement relations for *RRCCR* and othe spatial mechanisms.

Jenkins and Crossley[11], Sharma and Torfason[12], Dukkipati and Soni[13] used the method of generated surfaces to conduct analysis of single loop mechanisms containing revolute, prismatic, cylinder, helical and spheric pairs. Hartenberg and Denavit[14] contributed iterative techniques to conduct displacement analysis of spatial mechanisms using (4 × 4) matrices containing revolute, prismatic, cylinder, helical and spheric pairs. Uicker[15] explored in further detail the (4 × 4) matrix approach of Hartenberg and Denavit. Soni and Harrisberger[16] contributed an iterative approach for performing kinematic analysis using (3 × 3) matrices with dual elements. Kohli and Soni[17, 18] used finite screws to conduct displacement analysis of single-loop and two-loop space mechanisms involving *R*, *P*, *C*, *H* and *S* pairs.

Bagci[19] used a (3 × 3) screw matrix for displacement analysis of a mechanism containing two revo-

†Research Professor of Mechanical Engineering.
‡Associate Professor of Mechanical Engineering.
§Graduate Research Assistants in Mechanical Engineering.
¶*R*: revolute, *P*: prismatic, *C*: cylindric, *S*: spherical, *H*: helical, *Sp*: sphere–plane and *Sg*: sphere–groove joint.

lute pairs, one cylinder pair and one spheric pair. Dobrovolski[20] used the method of spherical images to analyze space mechanisms containing revolute and cylinder pairs. Duffy[21, 22], Duffy and Habib-Olahi[23] used the method of spherical triangles to derive displacement relations for five and six link mechanisms containing revolute and cylinder pairs. Keller[25] and Gupta[26] also analyzed space mechanisms containing revolute, prismatic, cylinder, helical and spheric pairs. Recently Kohli and Soni[26] and Singh and Kohli[27] used the method of pair constraint geometry and successive screw displacements to conduct analyses of single and multi-loop mechanisms.

In the present paper, screw displacements expressed in vector form and the pair geometry constraints, also expressed in vector form, are used to derive the displacement, velocity and acceleration equations for R–Sp–R, R–Sp–P, P–Sp–R, P–Sp–P and R–Sg–C three link mechanisms.

Since Revolute (R) and Prismatic (P) pairs are special cases of the cylinder pair (in prismatic pairs, the rotation is zero; for revolute pairs sliding is zero), we derive the analysis equation for C–Sp–C and C–Sg–C mechanisms, and then force rotations or translations at one or more pairs to zero, to obtain the equations for the above described three-link one degree of freedom mechanisms.

Briefly, the procedure for obtaining the analysis equations is as follows.

Step 1. Consider the C–Sp–C mechanism and the C–Sg–C mechanism.

Step 2. Separate the two moving links (Bodies 1 & 2) at the sphere–plane pair for the C–Sp–C case and at the sphere–grove pair for the C–Sg–C.

Step 3. Use the screw displacements in vector form to describe the new (jth) position of the sphere–plane (Sp) or sphere–groove (Sg) pairs from two sides of the pair.

Step 4. Use the pair geometry constraints on the position of the pair obtained from two sides.

Step 5. Force the cylindrical (C) joints as revolute (R) or prismatic (P) joints by setting the sliding or the rotation equal to zero at cylindrical pairs.

2. THE THREE-LINK MECHANISM AND ASSOCIATED VECTORS

Figure 1 shows the initial position of two rigid bodies grounded via cylindrical pairs and connected together by a sphere–plane pair. Also shown are the following vectors and scalar quantities:

$\mathbf{u}_A$ unit vector defining the direction of the axis of cylindric pair A.

$\mathbf{u}_B$ unit vector defining the direction of the axis of cylindrical pair B.

$\mathbf{P}$ vector locating the axis of cylindric pair at A in the fixed coordinate system.

$\mathbf{Q}$ vector locating the axis of cylindric pair at B in the fixed coordinate system.

$\mathbf{A}$ unit vector perpendicular to the plane of the Sp pair embedded in body 1.

$\mathbf{A}'$ vector embedded in body 2, congruent with $\mathbf{A}$ in the starting position, as shown in Fig. 1.

$\mathbf{R}$ vector locating point R, the sphere center in the fixed coordinate system.

θ_A rotation of link 1 about axis $\mathbf{u}_A$.

θ_B rotation of link 2 about axis $\mathbf{u}_B$.

S_A translation of link 1 along axis $\mathbf{u}_A$.

S_B translation of link 2 along axis $\mathbf{u}_B$.

Figure 2 shows the C–Sg–C mechanism with all associated vectors and scalars. Description of all parameters are the same as for the C–Sp–C mechanism except for the direction of the vector $\mathbf{A}$, which is now along the direction of the groove and also the addition of S_G, which is the translation of the sphere along the direction of $\mathbf{A}$.

3. PAIR GEOMETRY CONSTRAINT EQUATIONS

Figures 3 and 4 show a sphere–plane (Sp) pair and a sphere–groove (Sg) pair with the vector $\mathbf{R}$ locating R, the sphere center. The vector $\mathbf{A}$, in the Sp pair is defined as a vector perpendicular to the plane in which the sphere moves. In the Sg pair, the vector $\mathbf{A}$ defines the direction of the groove.

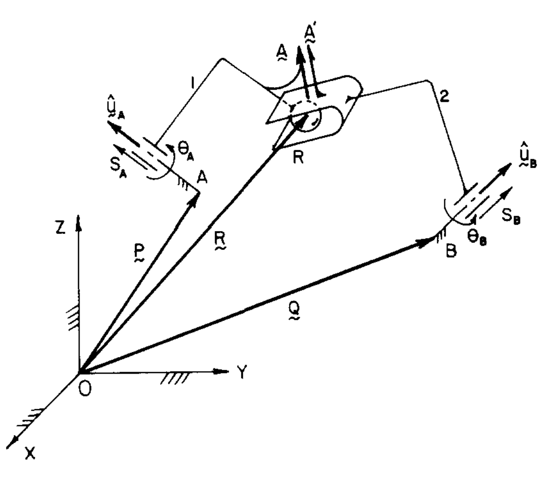

Fig. 1. C–S_p–C mechanism.

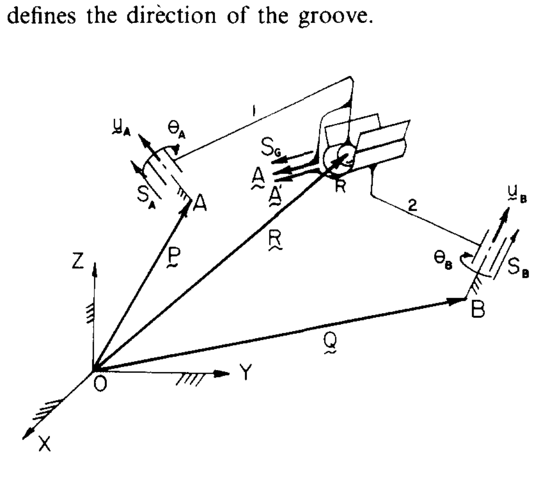

Fig. 2. C–S_g–C mechanism.

We can now define the vectors $\mathbf{R}_j$, $\mathbf{A}_j$, $\mathbf{R}'_j$ and $\mathbf{A}'_j$. These new vectors will define the displaced position and direction of initially coincident point R and vector $\mathbf{A}$ in bodies 1 and 2 respectively after some relative motion between bodies 1 and 2. The prime notation here is used for new position expressed from the motion of body 2, whereas the unprimed notations are used for new positions expressed from the motion of body 1.

The pair geometry constraint equation for the Sp pair is†

$$\frac{d^n}{dt^n}[(\mathbf{R}_j-\mathbf{R}'_j)\cdot\mathbf{A}'_j]=0,\ n=0,1,2,\ldots \quad (1)$$

which expresses that any relative motion between the sphere and the plane must be perpendicular to the vector $\mathbf{A}'_j$ (Fig. 1).

The pair geometry constraint equation for the Sg pair is

$$\frac{d^n\mathbf{R}_j}{dt^n}-\frac{d^n\mathbf{R}'_j}{dt^n}=\frac{d^n}{dt^n}(\mathbf{A}'_jS_{Gj})\ n=0,1,2,\ldots \quad (2)$$

where S_{Gj} is the translation of the sphere along the groove in the direction of $\mathbf{A}'_j$. The constraint equation for the Sg pair expresses that any relative motion between the sphere and the groove must be along the groove which is in the direction of $\mathbf{A}'_j$ (Fig. 2).

4. WORKING EQUATIONS

Referring to Fig. 1, let $\mathbf{A}$ be a vector in body 1 and $\mathbf{A}'$ a momentarily congruent vector in body 2 in the first position, perpendicular to the plane of the Sp pair. After some displacement of the mechanism, these vectors, in general, will separate due to the relative motion of the joint elements. Noting that both bodies 1 and 2 are connected to ground by C pairs, we use the equations developed by Kohli and Soni[26] for expressing the direction of a vector embedded in the rigid body and also the displaced position of a point of the body after a rotation θ about the cylinder axis and a translation S along the same axis. Using the prime notation for positions of the vector $\mathbf{A}'$ obtained from the motion of body 2 and the unprimed notation for positions of vector $\mathbf{A}$ (assumed frozen in body 1 in the first position and then moving with body 1) from the motion of body 1, the displaced directions of the vector $\mathbf{A}$ in bodies 1 and 2 are

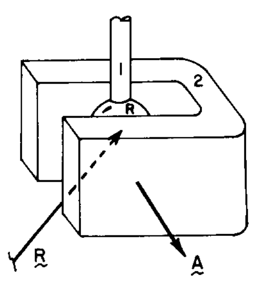

Fig. 3. Sphere–plane (S_p) pair.

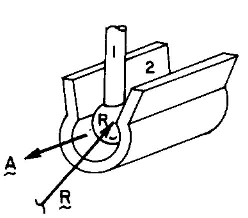

Fig. 4. Sphere–groove (S_g) pair.

†See Appendix for the derivation from the complete constraint equation.

$$\mathbf{A}_j=\cos\theta_{Aj}[\mathbf{A}-(\mathbf{A}\cdot\mathbf{u}_A)\mathbf{u}_A]+\sin\theta_{Aj}(\mathbf{u}_A\times\mathbf{A})$$
$$+(\mathbf{A}\cdot\mathbf{u}_A)\mathbf{u}_A \quad (3)$$

$$\mathbf{A}'_j=\cos\theta_{Bj}[\mathbf{A}-(\mathbf{A}\cdot\mathbf{u}_B)\mathbf{u}_B]+\sin\theta_{Bj}(\mathbf{u}_B\times\mathbf{A})$$
$$+(\mathbf{A}\cdot\mathbf{u}_B)\mathbf{u}_B \quad (4)$$

Also, the displaced position of the point R in rigid bodies 1 and 2 are given by:

$$\mathbf{R}_j=\cos\theta_{Aj}[(\mathbf{R}-\mathbf{P})-((\mathbf{R}-\mathbf{P})\cdot\mathbf{u}_A)\mathbf{u}_A]$$
$$+\sin\theta_{Aj}(\mathbf{u}_A\times(\mathbf{R}-\mathbf{P}))+[(\mathbf{R}-\mathbf{P})\cdot\mathbf{u}_A]\mathbf{u}_A$$
$$+\mathbf{u}_AS_{Aj}+\mathbf{P} \quad (5)$$

$$\mathbf{R}'_j=\cos\theta_{Bj}[(\mathbf{R}-\mathbf{Q})-((\mathbf{R}-\mathbf{Q})\cdot\mathbf{u}_B)\mathbf{u}_B]$$
$$+\sin\theta_{Bj}(\mathbf{u}_{Bj}(\mathbf{u}_B\times(\mathbf{R}-\mathbf{Q}))$$
$$+[(\mathbf{R}-\mathbf{Q})\cdot\mathbf{u}_B]\mathbf{u}_B+\mathbf{u}_BS_{Bj}+\mathbf{Q}. \quad (6)$$

Using the identity $[\mathbf{A}-(\mathbf{A}\cdot\mathbf{u}_A)\mathbf{u}_A]=(\mathbf{u}_A\times\mathbf{A})\times\mathbf{u}_A$, introducing the vectors

$$\mathbf{K}=\mathbf{R}-\mathbf{P}$$
$$\mathbf{L}=\mathbf{R}-\mathbf{Q} \quad (7)$$

and the following notation for any two vectors $\mathbf{u}_c$ and $\mathbf{D}$,

$$\mathbf{U}_{CD}=(\mathbf{u}_C\times\mathbf{D})\times\mathbf{u}_C, \quad (7a)$$

we can substitute eqns (7) and (7a) into eqns (5) and (6) to get

$$\mathbf{R}_j=\mathbf{R}+\mathbf{u}_AS_A+(\cos\theta_{Aj}-1)\mathbf{U}_{AK}+\sin\theta_{Aj}(\mathbf{u}_A\times\mathbf{K}) \quad (5a)$$

and

$$\mathbf{R}'_j=\mathbf{R}+\mathbf{u}_BS_B+(\cos\theta_{Bj}-1)\mathbf{U}_{BL}+\sin\theta_{Bj}(\mathbf{u}_B\times\mathbf{L}). \quad (6a)$$

We now take the time-derivatives of equations for $\mathbf{R}_j$ and $\mathbf{R}'_j$ and using the notation of dots above the variables to indicate time derivatives, we obtain the following equations

$$\dot{\mathbf{R}}_j = \mathbf{u}_A\dot{S}_{Aj} + [\cos\theta_{Aj}(\mathbf{u}_A \times \mathbf{K}) - \sin\theta_{Aj}\mathbf{U}_{AK}]\dot{\theta}_{Aj} \quad (8)$$

$$\dot{\mathbf{R}}'_j = \mathbf{u}_B\dot{S}_{Bj} + [\cos\theta_{Bj}(\mathbf{u}_B \times \mathbf{L}) - \sin\theta_{Bj}\mathbf{U}_{BL}]\dot{\theta}_{Bj} \quad (9)$$

$$\ddot{\mathbf{R}}_j = \mathbf{u}_A\ddot{S}_{Aj} - [\cos\theta_{Aj}\mathbf{U}_{AK} + \sin\theta_{Aj}(\mathbf{u}_A \times \mathbf{K})]\dot{\theta}^2_{Aj} + [\cos\theta_{Aj}(\mathbf{u}_A \times \mathbf{K}) - \sin\theta_{Aj}\mathbf{U}_{Ak}]\ddot{\theta}_{Aj} \quad (10)$$

$$\ddot{\mathbf{R}}'_j = \mathbf{u}_B\ddot{S}_{Bj} - [\cos\theta_{Bj}\mathbf{U}_{BL} + \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{L})]\dot{\theta}^2_{Bj} + [\cos\theta_{Bj}(\mathbf{u}_B \times \mathbf{L}) - \sin\theta_{Bj}\mathbf{U}_{BL}]\ddot{\theta}_{Bj}. \quad (11)$$

Substituting eqn (7a) into eqn (4), using eqns (5a) and (6a), and by making the following substitutions

$$\mathbf{M}_{Aj} = \cos\theta_{Aj}(\mathbf{u}_A \times \mathbf{K}) - \sin\theta_{Aj}\mathbf{U}_{AK}$$
$$\mathbf{M}_{Bj} = \cos\theta_{Bj}(\mathbf{u}_B \times \mathbf{L}) - \sin\theta_{Bj}\mathbf{U}_{BL} \quad (12)$$

$$\mathbf{N}_{Aj} = \cos\theta_{Aj}\mathbf{U}_{AK} + \sin\theta_{Aj}(\mathbf{u}_A \times \mathbf{K})$$
$$\mathbf{N}_{Bj} = \cos\theta_{Bj}\mathbf{U}_{BL} + \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{L}), \quad (13)$$

we can derive the following *working equations*

$$\mathbf{A}'_j = \mathbf{A} + (\cos\theta_{Bj} - 1)\mathbf{U}_{BA} + \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{A}) \quad (14)$$

$$\dot{\mathbf{A}}'_j = [\cos\theta_{Bj}(\mathbf{u}_B \times \mathbf{A}) - \sin\theta_{Bj}\mathbf{U}_{BA}]\dot{\theta}_{Bj} = \mathbf{V}_{Bj}\dot{\theta}_{Bj} \quad (15)$$

$$\ddot{\mathbf{A}}'_j = [\cos\theta_{Bj}(\mathbf{u}_B \times \mathbf{A}) - \sin\theta_{Bj}\mathbf{U}_{BA}]\ddot{\theta}_{Bj} - [\cos\theta_{Bj}\mathbf{U}_{BA} + \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{A})]\dot{\theta}^2_{Bj}$$

$$\ddot{\mathbf{A}}'_j = \mathbf{V}_{Bj}\ddot{\theta}_{Bj} - \mathbf{W}_{Bj}\dot{\theta}^2_{Bj} \quad (16)$$

where

$$\mathbf{V}_{Bj} = \cos\theta_{Bj}(\mathbf{u}_B \times \mathbf{A}) - \sin\theta_{Bj}\mathbf{U}_{BA}$$

and

$$\mathbf{W}_{Bj} = \cos\theta_{Bj}\mathbf{U}_{BA} + \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{A})$$

$$\mathbf{R}_j - \mathbf{R}'_j = \mathbf{u}_A S_{Aj} + (\cos\theta_{Aj} - 1)\mathbf{U}_{AK} + \sin\theta_{Aj}(\mathbf{u}_A \times \mathbf{K}) - \mathbf{u}_B S_{Bj} - (\cos\theta_{Bj} - 1)\mathbf{U}_{BL} - \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{L}) \quad (17)$$

$$\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j = \mathbf{u}_A\dot{S}_{Aj} + \mathbf{M}_{Aj}\dot{\theta}_{Aj} - \mathbf{u}_B\dot{S}_{Bj} - \mathbf{M}_{Bj}\dot{\theta}_{Bj} \quad (18)$$

$$\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}'_j = \mathbf{u}_A\ddot{S}_{Aj} - \mathbf{N}_{Aj}\dot{\theta}^2_{Aj} + \mathbf{M}_{Aj}\ddot{\theta}_{Aj} - \mathbf{u}_B\ddot{S}_{Bj} + \mathbf{N}_{Bj}\dot{\theta}^2_{Bj} - \mathbf{M}_{Bj}\ddot{\theta}_{Bj} \quad (19)$$

5. DISPLACEMENT ANALYSIS

To analyse the displacements of a particular 3-link one-degree-of-freedom mechanism containing either the *Sp* or *Sg* pair, we need only to take working eqn (17), apply the constraints of the particular grounded pairs and then substitute the results into the following pair geometry constraint equations for displacements.

For the *Sp* pair,

$$(\mathbf{R}_j - \mathbf{R}'_j)\cdot\mathbf{A}'_j = 0. \quad (20)$$

For the *Sg* pair,

$$(\mathbf{R}_j - \mathbf{R}'_j) = \mathbf{A}'_j S_{Gj}. \quad (21)$$

Observe that eqns (20) and (21) are eqns (1) and (2) with $n = 0$.

The cylindrical pairs used in the derivation may be forced to work as prismatic (P) pairs by letting $\theta \equiv 0$ or may be forced to work as revolute (R) pairs by letting $S \equiv 0$.

5.1 *The* P–Sp–P *case*

For this mechanism, we use $\theta_A \equiv \theta_B \equiv 0$ and eqns (14) and (17) are simplified to

$$\mathbf{R}_j - \mathbf{R}'_j = \mathbf{u}_A S_{Aj} - \mathbf{u}_B S_{Bj}$$

and

$$\mathbf{A}'_j = \mathbf{A}.$$

Substituting in eqn (20), we get

$$(\mathbf{u}_A S_{Aj} - \mathbf{u}_B S_{Bj})\cdot\mathbf{A} = 0 \quad (22)$$

which simplifies to the input/output equation

$$S_{Bj} = \frac{\mathbf{u}_A\cdot\mathbf{A}}{\mathbf{u}_B\cdot\mathbf{A}} S_{Aj}. \quad (23)$$

5.2 *The* R–Sp–P *case*

θ_A is the input; S_B is the output and $\theta_B \equiv S_A \equiv 0$. Equations (14) and (17) with $\theta_B \equiv S_A \equiv 0$ substituted in eqn (20) provide,

$$[-\mathbf{u}_B S_{Bj} + (\cos\theta_{Aj} - 1)\mathbf{U}_{AK} + \sin\theta_{Aj}(\mathbf{u}_A \times \mathbf{K})]\cdot\mathbf{A} = 0.$$

After simplification we obtain

$$S_{Bj} = \frac{[(\cos\theta_{Aj} - 1)\mathbf{U}_{AK} + \sin\theta_{Aj}(\mathbf{u}_A \times \mathbf{K})]\cdot\mathbf{A}}{\mathbf{u}_B\cdot\mathbf{A}}. \quad (24)$$

5.3 *The* R–Sp–R *case*

We have for this case $S_A \equiv S_B \equiv 0$, and eqn (14) and (17) are simplified to obtain

$$\mathbf{R}_j - \mathbf{R}'_j = (\cos\theta_{Aj} - 1)\mathbf{U}_{AK} + \sin\theta_{Aj}(\mathbf{u}_A \times \mathbf{K}) - (\cos\theta_{Bj} - 1)\mathbf{U}_{BL} - \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{L})$$

and

$$\mathbf{A}'_j = \mathbf{A} + (\cos\theta_{Bj} - 1)\mathbf{U}_{BA} + \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{A}).$$

Substituting the above equations into eqn (20), and simplifying the resulting equation, we obtain

$$-\mathbf{S}_j\cdot\mathbf{A} + (\cos\theta_{Bj} - 1)[-\mathbf{U}_{BL}\cdot\mathbf{A} - \mathbf{S}_j\cdot\mathbf{U}_{BA}] + \sin\theta_{Bj}[(\mathbf{u}_B \times \mathbf{L})\cdot\mathbf{A} - \mathbf{S}_j\cdot(\mathbf{u}_B \times \mathbf{A})] = 0 \quad (25)$$

where $\mathbf{S}_j$ is the known vector

$$\mathbf{S}_j = (\cos\theta_{Aj} - 1)\mathbf{U}_{AK} + \sin\theta_{Aj}(\mathbf{u}_A \times \mathbf{K}). \quad (26)$$

Equation (25) can be solved for θ_{Bj} by using the following identities

$$\cos\theta_B = \frac{1 - \tan^2\dfrac{\theta_B}{2}}{1 + \tan^2\dfrac{\theta_B}{2}}; \; \sin\theta_B = \frac{2\tan\dfrac{\theta_B}{2}}{1 + \tan^2\dfrac{\theta_B}{2}} \quad (27)$$

and simplifying the resulting quadratic equation to yield

$$\tan\frac{\theta_B}{2}j = \frac{-b \pm \sqrt{(b^2 - c(c-2a))}}{c - 2a} \quad (28)$$

where:

$$a = -\mathbf{U}_{BL}\cdot\mathbf{A} - \mathbf{S}_j\cdot\mathbf{U}_{BA}$$
$$b = (\mathbf{u}_B \times \mathbf{L})\cdot\mathbf{A} - \mathbf{S}_j\cdot(\mathbf{u}_B \times \mathbf{A})$$
$$c = -\mathbf{S}_j\cdot\mathbf{A}.$$

5.4 *The* P–Sp–R *case*

Here, $\theta_A \equiv S_B \equiv 0$ and we have

$$\mathbf{R}_j - \mathbf{R}'_j = \mathbf{u}_A S_{Aj} - 1(\cos\theta_{Bj} - 1)\mathbf{U}_{BL} - \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{L})$$

and

$$\mathbf{A}'_j = \mathbf{A} + (\cos\theta_{Bj} - 1)\mathbf{U}_{BA} + \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{L}).$$

Substituting the equations above into eqn (20) and simplifying, we get

$$(\cos\theta_{Bj} - 1)[-\mathbf{U}_{BL}\cdot\mathbf{A} - S_{Aj}(\mathbf{u}_A\cdot\mathbf{U}_{BA})]$$
$$+ \sin\theta_{Bj}[(\mathbf{u}_B\ \mathbf{L})\cdot\mathbf{A} - S_{Aj}\mathbf{u}_A\cdot(\mathbf{u}_B \times \mathbf{A})] - S_{Aj}\mathbf{u}_A\cdot\mathbf{A} = 0. \quad (29)$$

Substituting eqns (27) in eqn (29) and simplifying the resulting quadratic gives us

$$\tan\frac{\theta_B}{2}j = \frac{-b \pm \sqrt{(b^2 - c(c-2a))}}{c - 2a} \quad (30)$$

where this time

$$a = -\mathbf{U}_{BL}\cdot\mathbf{A} - S_{Aj}(\mathbf{u}_A\cdot\mathbf{U}_{BA})$$
$$b = (\mathbf{u}_B \times \mathbf{L})\cdot\mathbf{A} - S_{Aj}\mathbf{u}_A\cdot(\mathbf{u}_B \times \mathbf{A})$$
$$c = -S_{Aj}\mathbf{u}_A\cdot\mathbf{A}.$$

5.5 *The* R–Sg–C *case*

Only S_A in eqn (17) is identically zero, so we get

$$\mathbf{R}_j - \mathbf{R}'_j = -\mathbf{u}_B S_{Bj} + \mathbf{S}_j - (\cos\theta_{Bj} - 1)\mathbf{U}_{BL} - \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{L})$$

where $\mathbf{S}_j$ is given by eqn (26). Also,

$$\mathbf{A}'_j = \mathbf{A} + (\cos\theta_{Bj} - 1)\mathbf{U}_{BL} + \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{A}).$$

Substituting in eqn (21), we have

$$\mathbf{u}_B S_{Bj} - \mathbf{S}_j + (\cos\theta_{Bj} - 1)\mathbf{U}_{BL} + \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{L}) + \mathbf{A}'_j S_{Gj} = 0.$$

Forming the dot product of eqn (31) with $(\mathbf{A}'_j \times \mathbf{u}_B)$ and upon simplification, we get

$$\cos\theta_{Bj}[\mathbf{S}_j\cdot(\mathbf{A} \times \mathbf{u}_B) + \mathbf{U}_{BL}\cdot(\mathbf{A} \times \mathbf{u}_B)]$$
$$+ \sin\theta_{Bj}[\mathbf{S}_j\cdot\mathbf{U}_{BA} + \mathbf{U}_{BL}\cdot\mathbf{U}_{BA}] - (\mathbf{u}_B \times \mathbf{L})\cdot\mathbf{U}_{BA} = 0. \quad (32)$$

Again, θ_{Bj} can be obtained by substituting eqns (27) into eqn (32) to obtain a quadratic whose solutions are

$$\tan\frac{\theta_B}{2}j = \frac{-b \pm \sqrt{(a^2 + b^2 - c^2)}}{c - a} \quad (33)$$

where

$$a = \mathbf{S}_j\cdot(\mathbf{A} \times \mathbf{u}_B) + \mathbf{U}_{BL}\cdot(\mathbf{A} \times \mathbf{u}_B)$$
$$b = \mathbf{S}_j\cdot\mathbf{U}_{BA} + \mathbf{U}_{BL}\cdot\mathbf{U}_{BA}$$
$$c = -(\mathbf{u}_B \times \mathbf{L})\cdot\mathbf{U}_{BA}.$$

Forming the dot product of eqn (31) with $(\mathbf{u}_B \times \mathbf{L})$ and simplifying, we get

$$S_{Gj} = \frac{[\mathbf{S}_j\cdot(\mathbf{u}_B \times \mathbf{L}) - (\cos\theta_{Bj} - 1)\mathbf{U}_{BL}\cdot(\mathbf{u}_B \times \mathbf{L})]}{\mathbf{A}'_j\cdot(\mathbf{u}_B \times L)} - \frac{\sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{L})\cdot(\mathbf{u}_B \times \mathbf{L})}{\mathbf{A}'_j\cdot(\mathbf{u}_B \times \mathbf{L})}. \quad (34)$$

Forming the dot product of eqn (31) with $\mathbf{u}_B$ and simplifying, we get

$$S_{Bj} = [\mathbf{S}_j - (\cos\theta_{Bj} - 1)\mathbf{U}_{BL} - \sin\theta_{Bj}(\mathbf{u}_B \times \mathbf{L}) - S_{Gj}\mathbf{A}'_j]\cdot\mathbf{u}_B. \quad (35)$$

6. VELOCITY AND ACCELERATION ANALYSIS

To obtain the velocity and acceleration relations, we can either (a) take the derivatives with respect to time of the displacement equations or (b) use the higher order constraint equations. For the *P–Sp–P* case, taking the derivative of the displacement equation is trivial. But for the other cases, this procedure is cumbersome. It is therefore more convenient to just use eqns (14)–(19) in the following constraint eqns (36)–(39), which are eqns (1) and (2) with $n = 1$ and $n = 2$.

For the *Sp* pair

$$(\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j)\cdot\mathbf{A}'_j + (\mathbf{R}_j - \mathbf{R}'_j)\cdot\dot{\mathbf{A}}'_j = 0 \quad (36)$$

and

$$(\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}'_j)\cdot\mathbf{A}'_j + 2(\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j)\cdot\dot{\mathbf{A}}'_j + (\mathbf{R}_j - \mathbf{R}'_j)\cdot\ddot{\mathbf{A}}'_j = 0. \quad (37)$$

For the Sg pair

$$\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j = \mathbf{A}'_j\dot{S}_{Gj} + \dot{\mathbf{A}}'_jS_{Gj} \tag{38}$$

and

$$\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}'_j = \ddot{\mathbf{A}}'_jS_{Gj} + 2\dot{\mathbf{A}}'_j\dot{S}_{Gj} + \mathbf{A}'_j\ddot{S}_{Gj} \tag{39}$$

6.1 *The* P–Sp–P *case*

Here we can use the time derivatives of the displacement equation to get

$$S_{Bj} = \frac{\mathbf{u}_A\cdot\mathbf{A}}{\mathbf{u}_B\cdot\mathbf{A}}S_{Aj}$$

$$\dot{S}_{Bj} = \frac{\mathbf{u}_A\cdot\mathbf{A}}{\mathbf{u}_B\cdot\mathbf{A}}\dot{S}_{Aj} \tag{40}$$

$$\ddot{S}_{Bj} = \frac{\mathbf{u}_A\cdot\mathbf{A}}{\mathbf{u}_B\cdot\mathbf{A}}\ddot{S}_{Aj}. \tag{41}$$

6.2 *The* R–Sp–P *case*

Equations (18) and (19) become

$$\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j = \mathbf{M}_{Aj}\dot{\theta}_{Aj} - \mathbf{u}_B\dot{S}_{Bj}$$

and

$$\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}'_j = -\mathbf{N}_{Aj}\dot{\theta}^2_{Aj} + \mathbf{M}_{Aj}\ddot{\theta}_{Aj} - \mathbf{u}_B\ddot{S}_{Bj}$$

also

$$\mathbf{A}'_j = \mathbf{A}_j,\ \dot{\mathbf{A}}'_j = \ddot{\mathbf{A}}'_j = 0.$$

Substituting in eqns (36) and (37), we get

$$\dot{S}_{Bj} = \frac{\mathbf{M}_{Aj}\cdot\mathbf{A}}{\mathbf{u}_B\cdot\mathbf{A}}\dot{\theta}_{Aj} \tag{42}$$

and

$$\ddot{S}_{Bj} = -\frac{\mathbf{N}_{Aj}\cdot\mathbf{A}}{\mathbf{u}\cdot\mathbf{A}}\dot{\theta}^2_{Aj} + \frac{\mathbf{M}_{Aj}\cdot\mathbf{A}}{\mathbf{u}_B\cdot\mathbf{A}}\ddot{\theta}_{Aj}. \tag{43}$$

6.3 *The* R–Sp–R *case*

$$S_{Aj} \equiv S_{Bj} \equiv \dot{S}_{Aj} \equiv \dot{S}_{Bj} \equiv \ddot{S}_{Aj} \equiv \ddot{S}_{Bj} \equiv 0.$$

Equations (18) and (9) become

$$\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j = \mathbf{M}_{Aj}\dot{\theta}_{Aj} - \mathbf{M}_{Bj}\dot{\theta}_{Bj}$$

and

$$\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}'_j = -\mathbf{N}_{Aj}\dot{\theta}^2_{Aj} + \mathbf{M}_{Aj}\ddot{\theta}_{Aj} + \mathbf{N}_{Bj}\dot{\theta}^2_{Bj}\mathbf{M}_{Bj}\ddot{\theta}_{Bj}.$$

Also,

$$\dot{\mathbf{A}}'_j = \mathbf{V}_{Bj}\dot{\theta}_{Bj};\ \ddot{\mathbf{A}}'_j = \mathbf{V}_{Bj}\ddot{\theta}_{Bj} - \mathbf{W}_{Bj}\dot{\theta}^2_{Bj}.$$

Substituting in eqns (36) and (37), we get

$$\dot{\theta}_{Bj} = \frac{\mathbf{M}_{Aj}\cdot\mathbf{A}'_j}{\mathbf{M}_{Bj}\cdot\mathbf{A}'_j - (\mathbf{R}_j - \mathbf{R}'_j)\cdot\mathbf{V}_{Bj}}\dot{\theta}_{Aj} \tag{44}$$

and

$$\ddot{\theta}_{Bj} = \frac{-\mathbf{N}_{Aj}\cdot\mathbf{A}'_j}{D}\dot{\theta}^2_{Aj} + \frac{\mathbf{M}_{Aj}\cdot\mathbf{A}'_j}{D}\ddot{\theta}_{Aj} + \frac{2\mathbf{M}_{Aj}\cdot\mathbf{V}_{Bj}}{D}\dot{\theta}_{Aj}\dot{\theta}_{Bj} + \frac{\mathbf{N}_{Bj}\cdot\mathbf{A}'_j - 2\mathbf{M}_{Bj}\cdot\mathbf{V}_{Bj} - (\mathbf{R}_j - \mathbf{R}'_j)\cdot\mathbf{W}_{Bj}}{D}\dot{\theta}^2_{Bj} \tag{45}$$

where

$$D = \mathbf{M}_{Bj}\cdot\mathbf{A}'_j - (\mathbf{R}_j - \mathbf{R}'_j)\cdot\mathbf{V}_{Bj}. \tag{46}$$

6.4 *The* P–Sp–R *case*

Equations (18) and (19) are

$$\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j = \mathbf{u}_A\dot{S}_{Aj} - \mathbf{M}_{Bj}\dot{\theta}_{Bj}$$
$$\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}'_j = \mathbf{u}_A\ddot{S}_{Aj} + \mathbf{N}_{Bj}\dot{\theta}^2_{Bj} - \mathbf{M}_{Bj}\ddot{\theta}_{Bj}$$

also,

$$\dot{\mathbf{A}}'_j = \mathbf{V}_{Bj}\dot{\theta}_{Bj} \text{ and } \ddot{\mathbf{A}}'_j = \mathbf{V}_{Bj}\ddot{\theta}_{Bj} - \mathbf{W}_{Bj}\dot{\theta}^2_{Bj}.$$

Substituting in eqns (36) and (37) we get

$$\dot{\theta}_{Bj} = \frac{\mathbf{u}_A\cdot\mathbf{A}'_j}{\mathbf{M}_{Bj}\cdot\mathbf{A}'_j - (\mathbf{R}_j - \mathbf{R}'_j)\cdot\mathbf{V}_{Bj}}\dot{S}_{Aj} \tag{47}$$

and

$$\ddot{\theta}_{Bj} = \frac{1}{D}[\mathbf{u}_A\ddot{S}_{Aj} + 2(\mathbf{u}_A\cdot\mathbf{V}_{Bj})\dot{S}_{Aj}\dot{\theta}_{Bj} + (\mathbf{N}_{Bj}\cdot\mathbf{A}'_j - 2\mathbf{M}_{Bj}\cdot\mathbf{V}_{Bj} - (\mathbf{R}_j - \mathbf{R}'_j)\cdot\mathbf{W}_{Bj})\dot{\theta}^2_{Bj}] \tag{48}$$

where D is given by eqn (46).

6.5 *The* R–Sg–C *case*

Only S_{Aj}, $\dot{S}_{Aj}$ and $\ddot{S}_{Aj}$ are zero and eqns (18) and (19) become:

$$\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j = \mathbf{M}_{Aj}\dot{\theta}_{Aj} - \mathbf{u}_B\dot{S}_{Bj} - \mathbf{M}_{Bj}\dot{\theta}_{Bj}$$

and

$$\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}'_j = -\mathbf{N}_{Aj}\dot{\theta}^2_{Aj} + \mathbf{M}_{Aj}\ddot{\theta}_{Aj} - \mathbf{u}_B\ddot{S}_{Bj} + \mathbf{N}_{Bj}\dot{\theta}^2_{Bj} - \mathbf{M}_{Bj}\ddot{\theta}_{Bj}$$

also,

$$\dot{\mathbf{A}}'_j = \mathbf{V}_{Bj}\dot{\theta}_{Bj} \text{ and } \ddot{\mathbf{A}}'_j = \mathbf{V}_{Bj}\ddot{\theta}_{Bj} - \mathbf{W}_{Bj}\dot{\theta}^2_{Bj}.$$

Substituting the expression for $(\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j)$ just obtained into eqn (38) we get

$$\mathbf{M}_{Aj}\dot{\theta}_{Aj} - \mathbf{u}_B\dot{S}_{Bj} - \mathbf{M}_{Bj}\dot{\theta}_{Bj} = \mathbf{A}'_j\dot{S}_{Gj} + \mathbf{V}_{Bj}\dot{\theta}_{Bj}S_{Gj} \tag{49}$$

$\dot{\theta}_{Bj}$, $\dot{S}_{Bj}$ and $\dot{S}_{Bj}$ are unknowns in eqn (49).

Forming the dot product of eqn (49) with $(\mathbf{A}'_j \times \mathbf{u}_B)$, we get

$$(\mathbf{M}_{Aj}\dot{\theta}_{Aj} - \mathbf{M}_{Bj}\dot{\theta}_{Bj})\cdot(\mathbf{A}'_j \times \mathbf{u}_B) = \mathbf{V}_{Bj}\cdot(\mathbf{A}'_j \times \mathbf{u}_B)\dot{\theta}_{Bj}S_{Gj}$$

or

$$\dot{\theta}_{Bj} = \frac{\mathbf{M}_{Aj}\cdot\mathbf{A}'_j \times \mathbf{u}_B}{(S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj})\cdot\mathbf{A}'_j \times \mathbf{u}_B}\dot{\theta}_{Aj}. \tag{50}$$

Now, forming the dot product of eqn (49) with $[\mathbf{A}'_j \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj})]$, we have

$$(\mathbf{M}_{Aj}\dot{\theta}_{Aj} - \mathbf{u}_B\dot{S}_{Bj})\cdot\mathbf{A}'_j \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj}) = 0$$

or

$$\dot{S}_{Bj} = \frac{\mathbf{M}_{Aj} \cdot \mathbf{A}_j' \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj})}{\mathbf{u}_B \cdot \mathbf{A}_j' \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj})} \dot{\theta}_{Aj}. \tag{51}$$

Again forming the dot product of eqn (49) with

$$\mathbf{u}_B \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj}),$$

we have

$$(\mathbf{M}_{Aj}\dot{\theta}_{Aj} - \mathbf{A}_j'\dot{S}_{Gj}) \cdot \mathbf{u}_B \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj}) = 0$$

or

$$\dot{S}_{Gj} = \frac{\mathbf{M}_{Aj} \cdot \mathbf{u}_B \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj})}{\mathbf{A}_j' \cdot \mathbf{u}_B \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj})} \dot{\theta}_{Aj}. \tag{52}$$

Acceleration: Substituting the expression for $(\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}_j')$ obtained earlier for the R–Sg–C case into eqn (39), we will get

$$-\mathbf{N}_{Aj}\dot{\theta}_{Aj}^2 + \mathbf{M}_{Aj}\ddot{\theta}_{Aj} - \mathbf{u}_B\ddot{S}_{Bj} + \mathbf{N}_{Bj}\dot{\theta}_{Bj}^2 - \mathbf{M}_{Bj}\ddot{\theta}_{Bj} = (\mathbf{V}_{Bj}\ddot{\theta}_{Bj} - \mathbf{W}_{Bj}\dot{\theta}_{Bj}^2)S_{Gj} + 2\mathbf{V}_{Bj}\dot{\theta}_{Bj}\dot{S}_{Gj} + \mathbf{A}_j'\ddot{S}_{Gj}$$

or

$$\mathbf{u}_B\ddot{S}_{Bj} + \mathbf{A}_j'\ddot{S}_{Gj} + (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj})\ddot{\theta}_{Bj} = -\mathbf{N}_{Aj}\dot{\theta}_{Aj}^2 + \mathbf{M}_{Aj}\ddot{\theta}_{Aj} + (\mathbf{N}_{Bj} + S_{Gj}\mathbf{W}_{Bj})\dot{\theta}_{Bj}^2 - 2\mathbf{V}_{Bj}\dot{\theta}_{Bj}\dot{S}_{Gj}. \tag{53}$$

Letting $\mathbf{X}$ be equal to the r.h.s. of eqn (53) and by using the same technique of forming the dot product of eqn (53) with the proper cross-products, we will obtain the following

$$\ddot{\theta}_{Bj} = \frac{\mathbf{X} \cdot \mathbf{A}_j' \times \mathbf{u}_B}{(S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj}) \cdot \mathbf{A}_j' \times \mathbf{u}_B} \tag{54}$$

$$\ddot{S}_{Bj} = \frac{\mathbf{X} \cdot \mathbf{A}_j' \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj})}{\mathbf{u}_B \cdot \mathbf{A}_j' \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj})} \tag{55}$$

$$\ddot{S}_{Gj} = \frac{\mathbf{X} \cdot \mathbf{u}_B \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj})}{\mathbf{A}_j' \cdot \mathbf{u}_B \times (S_{Gj}\mathbf{V}_{Bj} + \mathbf{M}_{Bj})}. \tag{56}$$

7. NUMERICAL EXAMPLES

1. Analysis of a R–Sp–R mechanism.

The vectors describing the mechanism are

$$\mathbf{u}_A = 0\hat{\imath} + 1\hat{\jmath} + 0\hat{k}$$
$$\mathbf{u}_B = (3\hat{\imath} + 1\hat{\jmath} + 0\hat{k})/\sqrt{10}$$
$$\mathbf{P} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$
$$\mathbf{Q} = 0\hat{\imath} + 4\hat{\jmath} + 0.75\hat{k}$$
$$\mathbf{R} = 1\hat{\imath} + 1.5\hat{\jmath} + 2\hat{k}$$
$$\mathbf{A} = 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$$

The plot of the output displacement (θ_B), velocity ($\dot{\theta}_3$) and acceleration ($\ddot{\theta}_B$) are given in Fig. 5.

2. Displacement, velocity and acceleration analysis of a R–Sg–C mechanism.

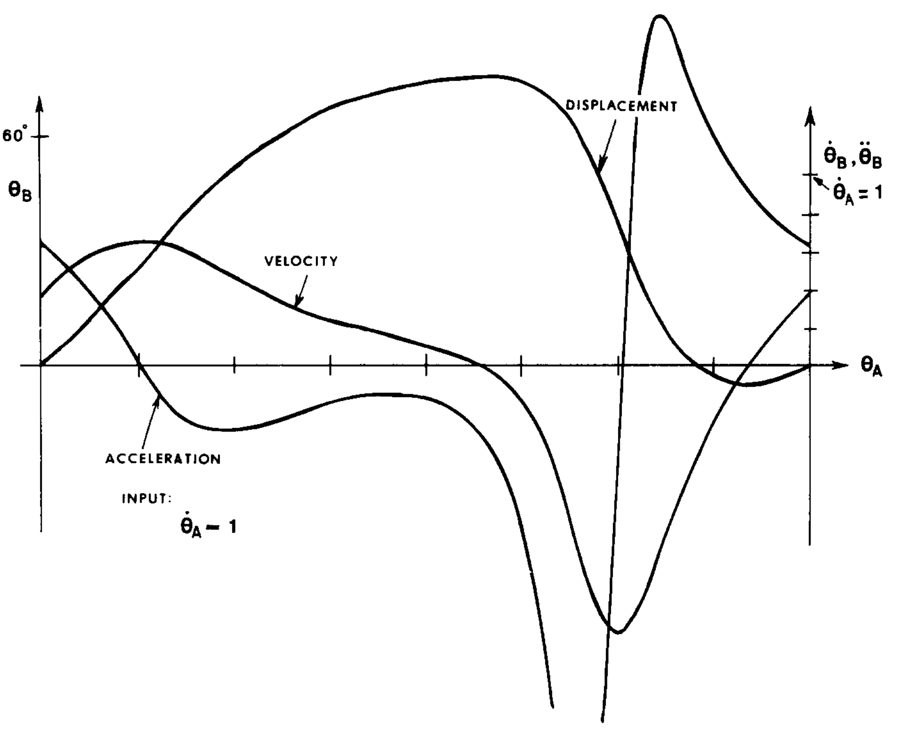

Fig. 5. Plot of θ_B, $\dot{\theta}_B$ and $\ddot{\theta}_B$ for the R–S_p–R mechanism.

Table 1. Displacements, velocities and accelerations

θ_A	θ_B	$\dot{\theta}_B$	$\ddot{\theta}_B$	S_G	$\dot{S}_G$	$\ddot{S}_G$	S_B	$\dot{S}_B$	$\ddot{S}_B$
5	2.12	.41	-.20	-.13	-1.56	-.75	.15	1.76	.36
40	12.99	.12	-1.23	-1.22	-1.95	-.41	1.26	1.79	-.32
80	-95.87	-3.36	32.3	-2.06	1.66	5.58	2.04	-.81	-4.07
120	-112.24	.26	.40	-.64	1.99	-.55	1.11	-1.59	-.43
160	-98.34	.39	.11	.58	1.42	-1.0	-.04	-1.64	.28
200	-81.04	.46	.08	1.32	.68	-1.06	-1.06	-1.20	.90
240	-61.47	.51	.06	1.55	.01	-.84	-1.65	-.44	1.23
280	-40.35	.54	.003	1.37	-0.5	-.67	-1.65	.44	1.24
320	-19.17	.51	-.07	.86	-.97	-.71	-1.05	1.23	.97
355	-2.2	.45	-.16	.13	-1.43	-.77	-.15	1.69	.52

The mechanism parameters are

$$\mathbf{u}_A = (1\hat{\imath} + 2\hat{\jmath} + 1\hat{k})/\sqrt{6}$$
$$\mathbf{u}_B = (1\hat{\imath} + 1\hat{\jmath} + 0\hat{k})/\sqrt{2}$$
$$\mathbf{P} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$
$$\mathbf{Q} = 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$$
$$\mathbf{R} = 3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$$
$$\mathbf{A} = (1\hat{\imath} + 1\hat{\jmath} + 2\hat{k})/\sqrt{6}.$$

The motion parameters are: $\dot{\theta}_{Aj}$ is one unit of angular velocity and $\ddot{\theta}_{Aj}$ is zero, both constant for $j = 0, 1, 2, \ldots$

The results of the analysis for the R–Sg–C mechanism are shown in Table 1.

The direction of the rotations and linear motions are established using the right hand rule. Rotations are positive counterclockwise looking at the head of the unit vectors $\mathbf{u}_A$ and $\mathbf{u}_B$. Linear motions are positive when they are in the direction of the vectors they are associated with.

It is to be mentioned here also that although the quadratic equations gave two sets of solutions, only one set will define the motion of the mechanism. The other set of solutions are for those positions in which the mechanism has to be disassembled into the other possible configuration.

8. CONCLUSIONS

Displacements, velocities and accelerations have been derived for several three-link spatial mechanisms containing sphere–plane and sphere–groove pairs. The groove of the sphere–groove pair was assumed to be a cylindrical groove, resulting in straight line axis of the groove. However, a more generalized groove may be one whose axis is a spatial curve. The authors are working on developing analysis procedures for mechanisms containing such a generalized sphere–groove pair. The expected results of their work will be the subject of a forthcoming paper. Similarly, the authors also have the generalization of the sphere–plane pair in progress, in which the parallel planes of the pair are generalized to form equidistant curved surfaces.

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APPENDIX

1. *Sphere–plane constraint equation*

The complete displacement constraint equations of the *Sp*-pair are

$$\mathbf{R}_j - \mathbf{R}'_j = S_{Pj}\mathbf{u}_{Pj} \tag{a}$$

and

$$\mathbf{u}'_{Pj} \cdot \mathbf{A}'_j = 0 \tag{b}$$

where $\mathbf{u}'_{Pj}$ is a unit vector in the plane of the *Sp* pair, perpendicular to $\mathbf{A}'_j$ and is in the direction of the relative motion of point R of body 1 with respect to the initially coincident point R' of body 2.

Derivatives of equations (a) and (b) with respect to time are taken to give the following velocity and acceleration constraint equations

Velocity:

$$\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j = \dot{S}_{Pj}\mathbf{u}'_{Pj} + S_{Pj}\dot{\mathbf{u}}'_{Pj} \tag{c}$$

and

$$\dot{\mathbf{u}}'_{Pj} \cdot \mathbf{A}'_j + \mathbf{u}'_{Pj} \cdot \dot{\mathbf{A}}_j = 0 \tag{d}$$

Acceleration:

$$\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}'_j = \ddot{S}_{Pj}\mathbf{u}'_{Pj} + 2\dot{S}_{Pj}\dot{\mathbf{u}}'_{Pj} + S_{Pj}\ddot{\mathbf{u}}'_{Pj} \tag{e}$$

and

$$\ddot{\mathbf{u}}'_{Pj} \cdot \mathbf{A}'_j + 2\dot{\mathbf{u}}'_{Pj} \cdot \dot{\mathbf{A}}'_j + \mathbf{u}'_{pj} \cdot \ddot{\mathbf{A}}'_j = 0. \tag{f}$$

The constraint eqns (a)–(f) are complete in the sense that all of the important variables in the motion of the joint elements are included. Also, the Coriolis component in the aceleration constraint eqn (f) is evident since $\dot{\mathbf{A}}'_j$ is a function of $\dot{\theta}_{Bj}$.

2. *Proof that* $(\mathrm{d}^n/\mathrm{d}t^n)[(\mathbf{R}_j - \mathbf{R}'_j) \cdot \mathbf{A}'_j] = 0$, $n = 0, 1, 2$ *satisfies the complete* Sp *pair constraint equation*

Without loss of generality, we can let $\mathbf{S}_{Pj} = S_{Pj}\mathbf{u}'_{Pj}$ and write the complete constraint equation as

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\mathbf{R}_j - \mathbf{R}'_j) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(\mathbf{S}_{Pj}) \tag{a}$$

and

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\mathbf{S}_{Pj} \cdot \mathbf{A}'_j) = 0. \tag{b}$$

Displacement: For $n = 0$, eqn (a) and (b) are

$$(\mathbf{R}_j - \mathbf{R}'_j) = \mathbf{S}_{Pj} \tag{c}$$

and

$$\mathbf{S}_{Pj} \cdot \mathbf{A}'_j = 0. \tag{d}$$

Forming the dot product of eqn (c) with $\mathbf{A}'_j$ gives us the displacement constraint equation for the *Sp* pair.

$$(\mathbf{R}_j - \mathbf{R}'_j) \cdot \mathbf{A}'_j = 0. \tag{e}$$

Velocity: With $n = 1$, eqns (a) and (b) will become

$$\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j = \dot{\mathbf{S}}_{Pj} \tag{f}$$

and

$$\dot{\mathbf{S}}_{Pj} \cdot \mathbf{A}'_j = -\mathbf{S}_{Pj} \cdot \dot{\mathbf{A}}'_j. \tag{g}$$

Taking the dot product of eqn (f) with $\mathbf{A}'_j$ gives us

$$(\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j) \cdot \mathbf{A}'_j = \dot{\mathbf{S}}_{Pj} \cdot \mathbf{A}'_j. \tag{h}$$

Substituting eqn (g) into (h), we will have

$$(\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j) \cdot \mathbf{A}'_j = -S_{Pj} \cdot \dot{\mathbf{A}}'_j. \tag{i}$$

Equation (c) can now be substituted in eqn (i) to get

$$(\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j) \cdot \mathbf{A}'_j = -(\mathbf{R}_j - \mathbf{R}'_j) \cdot \mathbf{A}'_j$$

or

$$(\dot{\mathbf{R}}_j - \dot{\mathbf{R}}'_j) \cdot \mathbf{A}'_j + (\mathbf{R}_j - \mathbf{R}'_j) \cdot \dot{\mathbf{A}}'_j = 0 \tag{j}$$

which is really

$$\frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{R}_j - \mathbf{R}'_j) \cdot \mathbf{A}'_j] = 0. \tag{k}$$

Acceleration: For $n = 2$, eqns (a) and (b) will be

$$\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}'_j = \ddot{\mathbf{S}}_{Pj} \tag{l}$$

and

$$\ddot{\mathbf{S}}_{Pj} \cdot \mathbf{A}_j' + 2\dot{\mathbf{S}}_{Pj} \cdot \mathbf{A}_j' + \mathbf{S}_{Pj} \cdot \ddot{\mathbf{A}}_j' = 0. \tag{m}$$

Forming the dot product of eqn (1) with $\mathbf{A}_j'$ and substituting

$$\ddot{\mathbf{S}}_{Pj} \cdot \mathbf{A}_j' = -2\dot{\mathbf{S}}_{Pj} \cdot \dot{\mathbf{A}}_j' - \mathbf{S}_{Pj} \cdot \ddot{\mathbf{A}}_j'$$

from eqn (m), we will get

$$(\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}_j') \cdot \mathbf{A}_j' = -2\dot{\mathbf{S}}_{Pj} \cdot \dot{\mathbf{A}}_j - \mathbf{S}_{Pj} \cdot \ddot{\mathbf{A}}_j'$$

or

$$(\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}_j') \cdot \mathbf{A}_j' + 2\dot{\mathbf{S}}_{Pj} \cdot \dot{\mathbf{A}}_j' + \mathbf{S}_{Pj} \cdot \ddot{\mathbf{A}}_j' = 0. \tag{n}$$

Substituting eqns (c) and (f) into eqn (n) gives us

$$(\ddot{\mathbf{R}}_j - \ddot{\mathbf{R}}_j') \cdot \mathbf{A}_j' + 2(\dot{\mathbf{R}}_j - \dot{\mathbf{R}}_j') \cdot \dot{\mathbf{A}}_j' + (\mathbf{R}_j - \mathbf{R}_j') \cdot \ddot{\mathbf{A}}_j' = 0$$

which is

$$\frac{d^2}{dt^2}[(\mathbf{R}_j - \mathbf{R}_j') \cdot \mathbf{A}_j'] = 0. \tag{o}$$

ANALYSE CINÉMATIQUE DES MÉCANISMES SPATIAUX A TROIS BARRES CONTENANT LES PAIRES SPHÈRE-PLAN ET SPHÈRE-RAINURE

G.N. Sandor, D. Kohli, M. Hernandez, Jr., A. Ghosal

Résumé - On considère généralement qu'une paire dans un mécanisme spatial permet un mouvement relatif de vis entre les membres, ou qu'elle restreint le mouvement des éléments qui lui sont reliés.

En employant les contraintes géométriques des paires de sphère-plan et de sphère-rainure cinématiques, les équations pour le déplacement, la vitesse et l'accélération sont dérivées pour les mécanismes avec trois membres R-Sp-R, R-Sp-P, P-Sp-P, P-Sp-R et R-Sr-C (R: révolute; P: prismatique; C: cylindrique; S: sphèrique; Sp: sphère-plan; Sr: sphère-rainure). Pour les valeurs connues de la variable d'entrée, les autres variables sont calculées par des formules non-itératives. Le procédé d'analyse est illustré par des exemples numériques.