Analytical determination of principal twists in serial, parallel and hybrid manipulators using dual vectors and matrices

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Abstract

The determination of principal twists of the end-effector of a multi-degree-of-freedom manipulator plays a central role in their analysis, design, motion planning and determination of singularities. Most approaches to obtain principal twists and the distributions of twists, such as the well-known classical results of cylindroid and hyperboloid, are based on geometric reasoning and involve intuitive choice of coordinate systems. In this paper, we present a formal algebraic approach to obtain the principal twists of any multi-degree-of-freedom serial, parallel or hybrid manipulator, by making use of the algebra of dual numbers, vectors and matrices. We present analytical expressions for the principal twists and the pitches for any arbitrary degree-of-freedom manipulator. A consequence of our approach is that we can obtain analytical expressions for the screws along which a manipulator can lose or gain degrees-of-freedom at a singularity. The theoretical results are illustrated with the help of examples of parallel and hybrid manipulators.

1 INTRODUCTION

It is well known in literature that rigid-body displacements constitute a group of isometries of $\Re^3$, known as the *Special Euclidean Group* (denoted by $SE(3)$). Twists, representing linear and angular velocity of a rigid body, lie in tangent-space at the identity of $SE(3)$, which is also the lie algebra (denoted by $se(3)$) associated with this group (see, e.g., [1]). Twists are 6-dimensional, but they may be expressed conveniently in terms of a pair of vectors in $\Re^3$. The central problem of screw theory is to find a suitable *principal* basis of $\Re^3$, in which the *basis vectors* of $se(3)$, or the *principal screws* take the simplest of forms. The principal screws are obtained using geometric arguments[2, 3]. In this paper, we make use of the dual orthogonal matrix representation of $SE(3)$[4] and arrive at the dual vector representation of twists. Dual 3-vectors form a free-module, $\mathcal{D}^3$, over the ring of dual numbers (denoted by Δ)[5], which is isomorphic to $se(3)$. This allows us to define an *inner product* on $se(3)$, and the corresponding *norm* of twists, as an element of Δ. Using these concepts, we reduce the problem of identification of principal twists to the extremization of the norm of the resultant twist. The extremization leads to an *eigen problem* of a symmetric dual matrix, consisting of the dual inner-products of the input screws, which is positive semi-definite over Δ. We show that our formulation results in analytical description of two different *principal bases*, which contain all the relevant informations about the first-order kinematics. Extremization of the real part of the dual norm leads to a basis (denoted by $\boldsymbol{\omega}-$basis), in which the degrees-of-freedom of a multi-degrees-of-freedom rigid-body motion decouples into pure translational and finite-pitch screw modes. We also show that the classical basis of Ball, which we denote as the $\boldsymbol{h}-$basis, and the $\boldsymbol{\omega}-$basis arise out of the general case of the dual eigen problem. We show that the determination of the principal screws of the $\boldsymbol{h}-$basis reduces to a generalized eigen problem involving the real and dual parts of a dual matrix. The principal pitches are proved to be half of the generalized eigenvalues, and analytical expressions for them can be obtained in closed form. We also show that singularities of loss and gain kind, in serial and parallel manipulators, can be treated naturally. These results form the most important contributions of the paper, and the theoretical development and the analytical results are more complete and elegant than the previous attempts of algebraic formulation of principal screws (see, e.g,[6, 7]).

The paper is organized as follows: in section 2, we present the notion of dual vectors and matrices and their use in representing twists. We present a symmetric dual matrix and discuss the eigen systems associated with it. In section 3, we present the analytical expressions of the principal twists for multi-degree-of-freedom rigid body motions and the concept of partitioning of degrees-of-freedom of a rigid body. In section 4, we present results on lost or gained twists at a singularity, and illustrate the theory, in section 5, with the help of a parallel and a hybrid manipulator.

2 MATHEMATICAL FORMULATION

The theoretical development and results presented in this paper are obtained using dual numbers, vectors and matrices. In this section, we first briefly review the notions of a dual number, vector and a matrix, and then apply them to analyze multi-degree-of-freedom rigid body motion.

2.1 Lines, Screws, and Twists as Dual Vectors

A dual number, $\hat{a}$, has the form $a + \epsilon a_0$, where $a,\ a_0 \in \Re$ and ϵ stands for the *dual unit*, with the properties $\epsilon \neq 0,\ \epsilon^2 = 0$. The properties of dual numbers are detailed in[8]. Here we note that the dual numbers over the real field form a *ring*, and dual n-vectors form a *free module* over this ring[5] denoted by $\mathcal{D}^n$. We can define an inner product on $\mathcal{D}^3$ as

$$\langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle = \boldsymbol{x} \cdot \boldsymbol{y} + \epsilon(\boldsymbol{x} \cdot \boldsymbol{y}_0 + \boldsymbol{y} \cdot \boldsymbol{x}_0) \tag{1}$$

The real and dual parts of the inner product are known to posses frame-invariance[1] and the dual inner product is positive semi-definite. A dual vector $\hat{\boldsymbol{x}}$ is called a *dual unit vector* if $\|\boldsymbol{x}\| = 1, \boldsymbol{x} \cdot \boldsymbol{x}_0 = 0$[8]. A line in $\Re^3$ can be described in terms of a dual unit vector as $\hat{\mathcal{L}} = \boldsymbol{Q} + \epsilon \boldsymbol{Q}_0$, where $(\boldsymbol{Q}; \boldsymbol{Q_0})$ is the Plücker vector associated with the line. The inner product of two lines, $\left\langle \hat{\mathcal{L}}_1, \hat{\mathcal{L}}_2 \right\rangle$, is given by $\cos\phi - \epsilon d \sin\phi = \cos\hat{\phi}$ where ϕ and d are the angle and the shortest distance between the two lines respectively[9]. A *screw* has five independent parameters and can be described by a dual vector $\hat{\boldsymbol{\$}} = \boldsymbol{Q} + \epsilon(\boldsymbol{Q}_0 + h\boldsymbol{Q}$. The pitch of the screw, h, is given by $\frac{\boldsymbol{S} \cdot \boldsymbol{S}_0}{\boldsymbol{S} \cdot \boldsymbol{S}}, \quad \|\boldsymbol{S}\| \neq 0$. If the magnitude of the real part of $\hat{\boldsymbol{\$}}$ is 0, and that of the dual part is non-zero, then the pitch is infinite, signifying a pure translation. The inner product of two screws is computed as $\cos\phi + \epsilon((h_1 + h_2)\cos\phi - d\sin\phi)$, where h_1 and h_2 are the pitches associated with the two screws respectively.

We parameterize $SE(3)$ in terms of dual orthogonal matrices of the form $\hat{\boldsymbol{A}} = \boldsymbol{R} + \epsilon \boldsymbol{D}\boldsymbol{R}$, where $\boldsymbol{R} \in SO(3)$ gives the orientation of the moving frame attached to the rigid-body with respect to some fixed reference frame, and $\boldsymbol{D} \in so(3)$ is the 3×3 skew-symmetric matrix associated with the displacement of the origin of the moving frame with respect to the fixed frame[4]. For n-DOF motions of the rigid-body, we can associate n independent real *motion parameters*, $\theta_i,\ i = 1, \ldots n$, via a smooth map, $\boldsymbol{\psi} : \Re^n \rightarrow SE(3)$ such that $\boldsymbol{\psi}(\boldsymbol{\theta}) = \hat{\boldsymbol{A}} \in SE(3)$. The motion parameters, $\boldsymbol{\theta}$, may be assumed to be functions of time t alone, and as $\boldsymbol{\theta}(t)$ evolves smoothly, it traces a *curve* $c(t) = \boldsymbol{\psi}(\boldsymbol{\theta}(t))$ on the manifold $SE(3)$, to each point of which we can associate a tangent space containing the velocity $\dot{c}(t)$ of the curve. The tangent vector $\dot{c}(t)$ may be obtained from the push-forward map $\boldsymbol{\psi}_* : \Re^n \rightarrow T_{\hat{\boldsymbol{A}}}SE(3)$ such that $\boldsymbol{\psi}_*(\dot{\boldsymbol{\theta}}) = \dot{\hat{\boldsymbol{A}}}(\boldsymbol{\theta}(t)) = \dot{\boldsymbol{R}} + \epsilon(\dot{\boldsymbol{D}}\boldsymbol{R} + \boldsymbol{D}\dot{\boldsymbol{R}}) \in T_{\hat{\boldsymbol{A}}}SE(3)$. We can translate this tangent vector to the tangent-space at the *identity* element of $SE(3)$ by left or right translations by $\hat{\boldsymbol{A}}^{-1} (= \hat{\boldsymbol{A}}^T)$ to obtain the Lie algebra $se(3)$ associated with the group, where the multiplication is given by the Lie bracket, denoted by $[\cdot, \cdot]$. The algebra $se(3)$ is isomorphic to the space of twists[1]. Depending upon the translation used to take them to the identity, we can get a *left-invariant* twist or a *right-invariant* twist. In this paper, we use the right-invariant twists[1], whose explicit form is $\hat{\boldsymbol{\Omega}} = \dot{\hat{\boldsymbol{A}}}\hat{\boldsymbol{A}}^T = \boldsymbol{\Omega} + \epsilon([\boldsymbol{D}, \boldsymbol{\Omega}] + \dot{\boldsymbol{D}})$ where $\boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^T \in so(3)$ denotes the right-invariant angular velocity of the rigid-body. Using the isomorphism of the algebras $(so(3), [\cdot,\cdot])$ and $(\Re^3, \times)$, we express the twist in terms of a dual vector, $\hat{\boldsymbol{\mathcal{V}}} = \boldsymbol{\omega} + \epsilon(\dot{\boldsymbol{d}} + \boldsymbol{d} \times \boldsymbol{\omega})$, where $\boldsymbol{\omega}, \dot{\boldsymbol{d}}$ and $\boldsymbol{d} \times \boldsymbol{\omega}$ are the counterparts of $\boldsymbol{\Omega}, \dot{\boldsymbol{D}}$, and $[\boldsymbol{D}, \boldsymbol{\Omega}]$ respectively in $\Re^3$. The quantity $\hat{\boldsymbol{\mathcal{V}}}$ is also known as a *motor*, and may be thought of as a screw together with a magnitude[9]. In terms of line coordinates, $\hat{\boldsymbol{\mathcal{V}}} = \|\boldsymbol{\omega}\|(\boldsymbol{Q} + \epsilon(\boldsymbol{Q}_0 + h\boldsymbol{Q}))$, where $\|\boldsymbol{\omega}\|$, the magnitude of the angular velocity vector, also denotes the magnitude of the twist.

The resultant twist of the end-effector of a n-DOF manipulator can be expressed as a linear combination of the input screws, $\hat{\$}_i, i = 1, ..., n$, as

$$\hat{\boldsymbol{\mathcal{V}}} = \sum_{i=1}^{n} \hat{\$}_i \dot{\theta}_i = \hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}} = \boldsymbol{J}_{\boldsymbol{\omega}}\dot{\boldsymbol{\theta}} + \epsilon \boldsymbol{J}_{\boldsymbol{v}}\dot{\boldsymbol{\theta}},\ i = 1, \ldots, n \tag{2}$$

where $\hat{\$}_i$, the ith column of $\hat{\boldsymbol{J}}$, may be computed as the vector form of the dual skew-symmetric matrix $\frac{\partial \hat{\boldsymbol{A}}}{\partial \theta_i}\hat{\boldsymbol{A}}^T$, and θ_i is the joint variable corresponding to the ith joint. The dual Jacobian, $\hat{\boldsymbol{J}}$, is composed of the Jacobians $\boldsymbol{J}_{\boldsymbol{\omega}}$ and $\boldsymbol{J}_{\boldsymbol{v}}$ corresponding to the angular and linear velocities respectively[2]. The square of the dual *norm* of $\hat{\boldsymbol{\mathcal{V}}}$ may be written as

$$\|\hat{\boldsymbol{\mathcal{V}}}\|^2 = \dot{\boldsymbol{\theta}}^T \hat{\boldsymbol{g}} \dot{\boldsymbol{\theta}} = \|\boldsymbol{\omega}\|^2(1 + \epsilon(2h)) \tag{3}$$

where $\|\boldsymbol{\omega}\|$ is the magnitude of the twist, and h is its pitch. The elements of the matrix $\hat{\boldsymbol{g}}$ are $\left\langle \hat{\$}_i, \hat{\$}_j \right\rangle$, and hence the matrix $\hat{\boldsymbol{g}}$ is *symmetric* and *frame invariant.*

Following the results for point trajectories[10], we seek the extremal values of the square of the magnitudes of the resultant twist, $\|\hat{\boldsymbol{\mathcal{V}}}\|^2$, subject to a *unit speed* constraint, $\|\dot{\boldsymbol{\theta}}\| = 1$. Using equation (3) and *Lagrange multipliers*, $\hat{\lambda}_i \in \Delta$, the objective function to be minimized is $\hat{g}_{ij}\dot{\theta}_i\dot{\theta}_j - \hat{\lambda}_i(\dot{\theta}_i^2 - 1), \quad i, j = 1, \ldots, n$. The solution of this n-dimensional extremization problem leads to the eigen problem

$$\hat{\boldsymbol{g}}\dot{\boldsymbol{\theta}} = \hat{\lambda}\dot{\boldsymbol{\theta}} \tag{4}$$

where $\hat{\lambda} = \lambda + \epsilon\lambda_0$ is the dual eigenvalue of $\hat{\boldsymbol{g}}$. In the following, we present the special properties of the eigen system of $\hat{\boldsymbol{g}}$ which form the basis of the theoretical results of this paper.

[1] Analogous results can be obtained for left-invariant twists.

[2] The Jacobians of parallel and hybrid manipulators can be obtained as shown in Appendix A.

2.2 Properties of the Eigensystem of $\hat{g}$

The ring of dual numbers, Δ, have the lexicographical order, namely $\hat{x}_1 >< \hat{x}_2$ if $x_1 >< x_2$, and if $x_1 = x_2$, then $\hat{x}_1 >=< \hat{x}_2$ if $x_{01} >=< x_{02}$. Therefore, the extremization of the magnitude of the resultant twist implies extremization of the real part of $\|\hat{\boldsymbol{\mathcal{V}}}\|^2$, i.e., $\|\boldsymbol{\omega}\|^2$. Expanding equation (4) into its real and dual parts, we get two real matrix equations,

$$\boldsymbol{g}\dot{\boldsymbol{\theta}} = \lambda\dot{\boldsymbol{\theta}}, \qquad \boldsymbol{g}_0\dot{\boldsymbol{\theta}} = \lambda_0\dot{\boldsymbol{\theta}} \tag{5}$$

Noting that $\boldsymbol{\omega} = \boldsymbol{J}_{\boldsymbol{\omega}}\dot{\boldsymbol{\theta}}$, it is easy to see from equations (3) that the extremization of $\|\boldsymbol{\omega}\|^2$ reduces to the first of equations (5) under the constraint $\|\dot{\boldsymbol{\theta}}\| = 1$. However, if we consider the space of screws alone, with $\boldsymbol{\omega} = 1$, then as per the lexicographical order in Δ, $\|\hat{\boldsymbol{\mathcal{V}}}\|^2$ is extremized when its dual part is extremized. The dual part is twice the pitch, h, and can be shown to be equal to $h = \frac{\lambda_0}{2\lambda}$. This leads to two bases discussed in the following sections.

2.3 Formulation of the $\boldsymbol{\omega}-$basis

Since $rank_{\Re}\boldsymbol{J}_{\boldsymbol{\omega}} \leq 3$, the characteristic polynomial of $\boldsymbol{g}$, for n-DOF motion ($n \geq 3$), reduces to the form

$$\lambda^{n-3}(\lambda^3 - a_{n-1}\lambda^2 + a_{n-2}\lambda + a_{n-3}) = 0 \tag{6}$$

and hence we have to solve for at most a cubic. The cubic is guaranteed to have real roots, since $\boldsymbol{g}$ is symmetric and the cubic can be solved *analytically* in closed form using Cardan's formula. Hence, we can obtain analytical expressions for the eigenvalues and the eigenvectors in terms of the input screw parameters. The eigenvectors, denoted by $\dot{\boldsymbol{\theta}}_i^{\omega}$, form a basis of the row-space of $\hat{\boldsymbol{J}}$, and the principal twists, denoted by $\hat{\boldsymbol{\mathcal{V}}}_i^{\omega}$ lying in the column-space of $\hat{\boldsymbol{J}}$ are $\hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}}_i^{\omega}$. The set $\{\hat{\boldsymbol{\mathcal{V}}}_i^{\omega}\}, \quad i = 1, \cdots, n$ constitute the $\boldsymbol{\omega}-$basis.

2.4 Formulation of the $\boldsymbol{h}-$basis

It is known from linear algebra, that the two matrices $\boldsymbol{g}$ and $\boldsymbol{g}_0$ share an eigenvector, $\dot{\boldsymbol{\theta}}^h$, iff $\boldsymbol{g}\boldsymbol{g}_0 = \boldsymbol{g}_0\boldsymbol{g}$. We show that if $\boldsymbol{g}$ is positive definite we can always find a transformation T of $\Re^n$, which will reduce $\boldsymbol{g}$ and $\boldsymbol{g}_0$ to such forms that they commute. From equation (6), we can observe that the matrix $\boldsymbol{g}$ is positive definite when the $n \leq 3$ - for $n \geq 3$, $n-3$ eigenvalues are zero. For $n \leq 3$, the required transformation may be obtained in three steps:

1. *Diagonalization of $\boldsymbol{g}$ and transformation of $\boldsymbol{g}_0$:* The transformation $\boldsymbol{T}_1$ has the eigenvectors of $\boldsymbol{g}$ as its columns and we can write

$$\boldsymbol{g}_1 = \boldsymbol{T}_1^{-1}\boldsymbol{g}\boldsymbol{T}_1 = \mathrm{diag}\{\lambda_i\},\ i = 1,\dots,n, \qquad \boldsymbol{g}_{01} = \boldsymbol{T}_1^{-1}\boldsymbol{g}_0\boldsymbol{T}_1$$

2. *Scaling by the square-root of λ_i:* The transformation T_2 is given by diag $\{1/\sqrt{\lambda_i}\},\ i = 1,\dots,n$, and we get

$$\boldsymbol{g}_2 = \boldsymbol{T}_2^{-1}\boldsymbol{g}_1\boldsymbol{T}_2 = \mathrm{diag}\{1,\dots,1\}, \qquad \boldsymbol{g}_{02} = \boldsymbol{T}_2^{-1}\boldsymbol{g}_{01}\boldsymbol{T}_2$$

3. *Diagonalization of $\boldsymbol{g}_{02}$ and transformation of $\boldsymbol{g}_2$:* The transformation $\boldsymbol{T}_3$ has the eigenvectors of $\boldsymbol{g}_{02}$ as its columns and we get

$$\boldsymbol{g}_3 = \boldsymbol{T}_3^{-1}\boldsymbol{g}_2\boldsymbol{T}_3 = \mathrm{diag}\{1,\dots,1\},\ \ \boldsymbol{g}_{03} = \boldsymbol{T}_3^{-1}\boldsymbol{g}_{02}\boldsymbol{T}_3 = \mathrm{diag}\{2h_i\},\ i = 1,\dots,n.$$

The total transformation, $\boldsymbol{T} = \boldsymbol{T}_1\boldsymbol{T}_2\boldsymbol{T}_3$, reduces $\boldsymbol{g}$ into an identity matrix and $\boldsymbol{g}_0$ to a diagonal matrix with entries $2h_i$. The eigenvectors, $\dot{\boldsymbol{\theta}}^h$, obtained by solving the generalized eigen problem $\boldsymbol{g}_0\dot{\boldsymbol{\theta}}^h = 2h_i\boldsymbol{g}$, when mapped by $\hat{\boldsymbol{J}}$ leads to *principal screws*, and $\hat{\$}_i^h = \hat{\boldsymbol{J}}\dot{\boldsymbol{\theta}}_i^h$ constitute the $\boldsymbol{h}-$basis. The matrices $\boldsymbol{g}$ and $\boldsymbol{g}_0$ are diagonal in this basis, and the principal screws *meet at one point in space orthogonally*, and as proved above, their pitches are extremal. These two observations identify the $\boldsymbol{h}-$basis as the *classical* principal basis as described in [2, 3]. It may be noted, that as in the $\boldsymbol{\omega}-$basis, the generalized eigenvalues and eigenvectors can be obtained analytically, since we have to solve at most a cubic equation.

3 PRINCIPAL TWISTS IN $\boldsymbol{\omega}-$basis

We now present the analytical expressions of principal twists in the $\boldsymbol{\omega}-$basisfor multi-DOF rigid-body motionand partitioning of degrees-of-freedom. We present the results for various degrees-of-freedom[3].

One-degree-of-freedom rigid body motion: In this case, the distribution of allowable twists is of the form $\hat{\boldsymbol{\mathcal{V}}} = \hat{\$}_1\dot{\theta}_1$. The single input screw $\hat{\$}_1$ is the the principal screw of the system, and transforming to a frame where the $\boldsymbol{X}$ axis is along the screw axis, and the origin is some chosen point on the axis, the principal twist is $k(1+\epsilon h^*)(1,0,0)^T$ where h^* is the pitch of $\hat{\$}$ and $k \in \Re$ is the magnitude of the input, assumed to be unity under the unit-speed constraint.

Two-degrees-of-freedom rigid body motion : Let $\boldsymbol{\theta}(t) = (\theta_1(t), \theta_2(t))^T$ represent the two independent motion parameters and $\hat{\$}_i,\ i = 1,2$ represent the two

[3]The following results are directly applicable to serial manipulators. For parallel and hybrid manipulators, similar analytical expressions can be written in terms of g_{ij} and $g_{0_{ij}}$.

input screws. The resultant twist, $\hat{\mathcal{V}}$ is $\hat{\$}_1\dot{\theta}_1 + \hat{\$}_2\dot{\theta}_2$. The elements of $\hat{\boldsymbol{g}}$ are[4] are $c_{ij} + \epsilon((h_i + h_j)c_{ij} - d_{ij}s_{ij})$, $i, j = 1, 2$ with $c_{ii} = 1$ and $s_{ii} = 0$. The real and dual part of the characteristic equation can be solved to obtain

$$\begin{aligned}\hat{\lambda}_1 &= 2\cos^2\phi_{12}/2(1 + \epsilon(h_1 + h_2 - d_{12}\tan(\phi_{12}/2))) \\ \hat{\lambda}_2 &= 2\sin^2\phi_{12}/2(1 + \epsilon(h_1 + h_2 + d_{12}\cot(\phi_{12}/2)))\end{aligned} \tag{7}$$

The principal magnitude and pitches are given by

$$\begin{aligned}\|\boldsymbol{\omega}_1^{\boldsymbol{\omega}}\| &= \sqrt{2}\cos\phi_{12}/2, & \|\boldsymbol{\omega}_2^{\boldsymbol{\omega}}\| &= \sqrt{2}\sin\phi_{12}/2 \\ h_1^{\boldsymbol{\omega}} &= 1/2(h_1 + h_2 - d_{12}\tan(\phi_{12}/2)), & h_2^{\boldsymbol{\omega}} &= 1/2(h_1 + h_2 + d_{12}\cot(\phi_{12}/2))\end{aligned} \tag{8}$$

The real eigenvectors of $\hat{\boldsymbol{g}}$ are given by $1/\sqrt{2}(1 \pm 1)^T$, and they map to the principal twists as $\frac{1}{\sqrt{2}}(\hat{\$}_1 \pm \hat{\$}_2)$.

Three-degrees-of-freedom rigid body motion: In this case, the resultant twist is $\hat{\$}_1\dot{\theta}_1 + \hat{\$}_2\dot{\theta}_2 + \hat{\$}_3\dot{\theta}_3$. The characteristic polynomial of $\hat{\boldsymbol{g}}$ can be written as

$$\lambda^3 - 3\lambda^2 + (3 - c_{12}^2 - c_{23}^2 - c_{31}^2)\lambda + (c_{12}^2 + c_{23}^2 + c_{31}^2 - 2c_{12}c_{23}c_{31} - 1) = 0 \tag{9}$$

which has real roots

$$\lambda_i = 1 + \frac{2\sqrt{3}}{3}\sqrt{c_{12}^2 + c_{23}^2 + c_{31}^2}\cos\left(\frac{\phi + (i-1)2\pi}{3}\right) \tag{10}$$

where $i = 1, 2, 3$, and $\phi \in [0, 2\pi]$ is such that $\sin\phi = (1/27)(c_{12}^2 + c_{23}^2 + c_{31}^2)^3 - c_{12}^2c_{23}^2c_{31}^2$ and $\cos\phi = c_{12}c_{23}c_{31}$. The principal magnitudes and pitches are $\|\boldsymbol{\omega}_i^{\boldsymbol{\omega}}\| = \sqrt{\lambda_i}$, $i = 1, 2, 3$, and

$$h_i^{\boldsymbol{\omega}} = -\frac{a_2\lambda_i^2 + a_1\lambda_i + a_0}{3\lambda_i^3 - 6\lambda_i^2 + (3 - (c_{12}^2 + c_{23}^2 + c_{31}^2))\lambda_i} \tag{11}$$

Using the notation $H = h_1 + h_2 + h_3$, where h_i, $i = 1, 2, 3$ are the pitches of the input screws, we have

$$\begin{aligned}a_2 = -2H, \qquad a_1 &= H(2 - c_{12} - c_{23} - c_{31}) + h_1c_{23} + h_2c_{23} + h_3c_{12} \\ a_0 &= H(\cos_{12} + \cos_{23} + \cos_{31} - 4c_{12}c_{23}c_{31}) + 2d_{12}(c_{23}c_{31} - c_{12}) \\ &+ 2d_{31}(c_{12}c_{23} - c_{31}) + 2d_{23}(c_{12}c_{31} - c_{23})\end{aligned} \tag{12}$$

The ith eigenvector of $\hat{\boldsymbol{g}}$ is given by $\dot{\boldsymbol{\theta}}_i = \left(\frac{c_{12}c_{31} + c_{23}(1+\lambda_i)}{(1+\lambda_i)^2 - c_{12}^2}, \frac{c_{12}c_{23} + c_{31}(1+\lambda_i)}{(1+\lambda_i)^2 - c_{12}^2}, 1\right)^T$, and they can be mapped by the dual Jacobian to obtain the principal twists $\hat{\mathcal{V}}_i^{\boldsymbol{\omega}}$. It may

[4]We use d_{ij} and ϕ_{ij} to denote the distance and angle between the ith and jth screw axes, and c_{ij} and s_{ij} to denote $\cos\phi_{ij}$ and $\sin\phi_{ij}$ respectively.

be verified that for distinct λ_i, the axes of the principal twists are orthogonal, but, in general, they do not coincide in $\Re^3$.

Rigid-body motion with DOF > 3 : The general case of n-DOF motion can be considered within the same framework by noting that the $rank_{\Re}(\boldsymbol{J_\omega}) \leq 3$, and hence $rank_{\Delta}(\hat{\boldsymbol{g}}) \leq 3$, which restricts the characteristic polynomial of $\hat{\boldsymbol{g}}$ to at most a dual cubic. More explicitly, the characteristic equation takes the form

$$\hat{\lambda}^{n-3}(\hat{\lambda}^3 + \hat{a}_{n-1}\hat{\lambda}^2 + \hat{a}_{n-2}\hat{\lambda} + \hat{a}_{n-3}) = 0 \tag{13}$$

We conclude from the above that $n-3$ of the eigenvalues are zeros, and the 3 non-zero ones can be computed from the residual cubic equation, once the coefficients are computed from the dual invariants of $\hat{\boldsymbol{g}}$. We also note that $a_{n-1} = -n$, as it is the negative of the trace of $\boldsymbol{g}$ and $\hat{\boldsymbol{g}}_{ii} = 1 + \epsilon(2h_i)$. The residual cubic equation, $\lambda^3 - n\lambda^2 + a_{n-2}\lambda + a_{n-3} = 0$, requires the computation of only two coefficients, which are the second and the third invariants of $\boldsymbol{g}$. Thus, by exploiting the algebraic structure of the problem, we ensure an analytic solution for rigid-body motion of arbitrary DOF greater than 3. The 3 eigenvectors corresponding to the non-zero eigenvalues and $n-3$ principal twists in the null-space of $\hat{\boldsymbol{J}}$ can be computed by using standard linear algebra methods.

3.1 Partitioning of DOF

If $nullity(\boldsymbol{J_\omega}) = m,\ m \in \mathcal{Z}^+,\ m \neq 0$, then m of the principal twists will lie in the left null-space of $\hat{\boldsymbol{J}}$. Expressed as dual vectors, these twists are of the form $\boldsymbol{0} + \epsilon \boldsymbol{v}_i^{\boldsymbol{\omega}} = \epsilon \boldsymbol{J_v} \dot{\boldsymbol{\theta}}_i, \quad (i = 1, \ldots, n-3)$ for n-DOF motion ($n > 3$). These twists have infinite pitches, and they signify *pure translational motion* of the rigid-body. Rigid-body motion can thus be divided into two parts, namely, one consisting of both rotation and translation (finite-pitch motion), and another consisting of purely translational motion, and independent of the rotational motion of the rigid-body. This *DOF partitioning* allows us to study the rotational and translational modes of rigid-body motion independent of each other, and our *analytical* expressions for the principal twists can now be profitably used for robotic applications where the end-effector motion requirements can be split into these two modes explicitly.

4 ANALYSIS OF SINGULARITIES IN $\boldsymbol{\omega}$−basis

The analytical expressions of the principal twists, derived in the previous section, can be profitably used for analysis of singularities. In this section, we discuss both the loss and gain kinds of singularities seen in serial and parallel manipulators[11].

4.1 Loss Type of Singularity

The loss kind of singularity is said to occur when the manipulator end-effector fails to twist about certain screw(s) in spite of full actuation. This results in the loss of one or more degrees-of freedom of the end-effector[12]. We first consider loss of rotational DOF.

The manipulator end-effector has $1, 2$ or 3 rotational degrees-of-freedom depending upon the number of non-zero eigenvalues $\hat{\boldsymbol{g}}$ has at a non-singular configuration. If at a singular configuration, m additional eigenvalues vanish[5], then we say that the manipulator has lost m rotational degrees-of-freedom. It may be noted that the corresponding pitch also vanishes, and hence the corresponding twist reduces to a pure translation in the null-space of $\hat{\boldsymbol{J}}$ at that configuration. We look at the possibilities on a case by case basis.

One-degree-of-freedom: In this case, the principal screw reduces to a null vector, $\mathbf{0} + \epsilon\mathbf{0}$, unless the original DOF was translational (as in a P-joint), in which case there is no loss of rotational DOF possible.

Two-degrees-of-freedom: From the set of equations (7), $\hat{\lambda}_2$ can vanish if $\sin^2 \phi_{12} = 0$. The two principal twists collapse to $\hat{\boldsymbol{\mathcal{V}}}_1^{\boldsymbol{\omega}} = \frac{1}{\sqrt{2}}(\hat{\$}_1 + \hat{\$}_2)$ which gives the resultant rotational DOF in this case, and $\hat{\boldsymbol{\mathcal{V}}}_2^{\boldsymbol{\omega}} = \frac{1}{\sqrt{2}}(\hat{\$}_1 - \hat{\$}_2)$, now forms the left null-space of $\hat{\boldsymbol{J}}$, signifying a translatory DOF in addition to the residual rotational DOF.

Three-degrees-of-freedom: In this case, there may be loss of one or two angular degrees-of-freedom, the conditions of the same are found from equation (9) as $c_{12}^2 + c_{23}^2 + c_{31}^2 - 2c_{12}c_{23}c_{31} - 1 = 0$ and $c_{12}^2 + c_{23}^2 + c_{31}^2 - 2c_{12}c_{23}c_{31} - 1 = 0 = (3 - c_{12}^2 - c_{23}^2 - c_{31}^2)$ respectively. The non-zero roots may be computed from equation (9), which reduces to a quadratic and a linear equation in λ in the two cases respectively. The eigenvectors of $\boldsymbol{g}$ can be computed symbolically, and therefrom the principal twists in the column-space and null space of $\hat{\boldsymbol{J}}$ can be obtained.

Degrees-of-freedom(n)> 3: The treatment in this case follows exactly the case of three-degrees-of-freedom. We need to consider the general equation and the conditions for loss of one or two rotational DOF are $a_{n-3} = 0$, and $a_{n-3} = 0 = a_{n-2}$ respectively.

The number of pure translational degrees-of-freedom equal the number of linearly independent pure dual vectors in the left null space of $\hat{\boldsymbol{J}}$ and they span the space of pure translational velocities of the rigid body. We write their dual parts as the columns of a $3 \times m$ real matrix, $\mathbf{B}$, and let the rank of $\mathbf{B}$ be $r (r \leq 3)$. At a singularity leading to loss of translational DOF, the rank of $\mathbf{B}$ reduces by 1, 2 or 3. It may be

[5] m can be either 1 or 2. All the three eigenvalues can vanish only for a purely Cartesian manipulator, whose analysis can be done much more conveniently by looking at its linear velocity distribution in $\Re^3$.

noted that loss of rotational motion also leads to the addition of a column to **B**, but since the rank of **B** is limited to 3, the degeneracy of rotational motion does not lead to an additional translational DOF if rank of **B** is already 3.

4.2 Gain Type of Singularity

A parallel device gains one or more degrees-of-freedom in the configuration space when one of the constraint Jacobians, $\boldsymbol{J_{\eta\phi}}$, loses rank (see Appendix A for derivation of Jacobians for parallel and hybrid manipulators), and the number of DOF gained equals the nullity of $\boldsymbol{J_{\eta\phi}}$ (see, for example, [13]). The *gained* passive motions lie in the null-space of $\boldsymbol{J_{\eta\phi}}$, and may be obtained by solving $\boldsymbol{J_{\eta\phi}}\dot{\boldsymbol{\phi}}_i = \mathbf{0}$, $i = 1, \ldots, nullity(\boldsymbol{J_{\eta\phi}})$. The effect of this gain is that the manipulator end-effector can now twist about one or more screws even with all the actuators locked. These twists are given by

$$\hat{\boldsymbol{\mathcal{V}}}_i = \boldsymbol{J_{\omega\phi}}\dot{\boldsymbol{\phi}}_i + \epsilon \boldsymbol{J_{v\phi}}\dot{\boldsymbol{\phi}}_i \tag{14}$$

The gained principal twists can be obtained analytically since, once again, we need to solve at the most a cubic equation.

5 ILLUSTRATIVE EXAMPLES

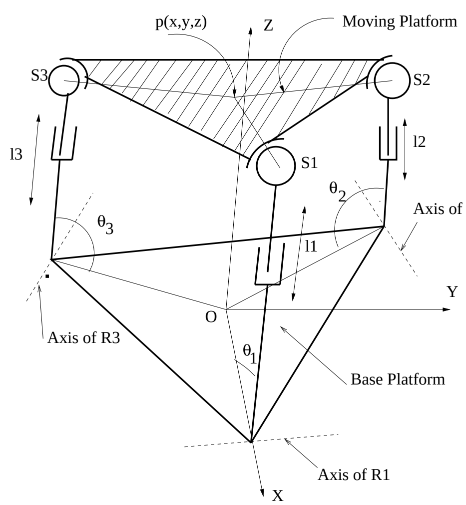

Figure 1: The 3-RPS Parallel Manipulator

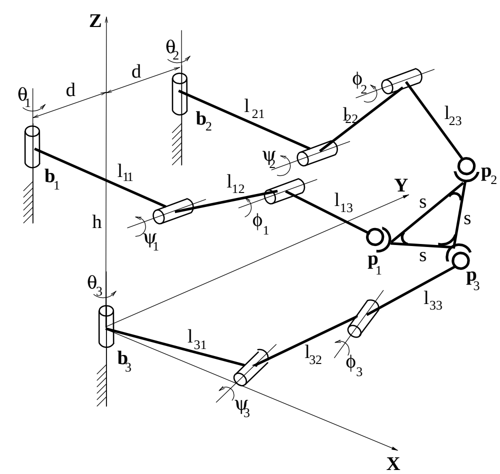

Figure 2: The 6-DOF Hybrid Manipulator

The above developed theory is illustrated by an examples of a 3-DOF parallel manipulator shown in figure 1 and a 6-DOF hybrid manipulator shown in figure 2.

At a non-singular configuration for the 3-DOF parallel manipulator defined by $l_1 = 1$, $l_2 = 2/3$, $l_3 = 3/4$, and corresponding passive variables $\theta_1 = 0.878516$ rad , $\theta_2 = 0.905239$ rad and $\theta_3 = 0.120906$ rad, the dual eigenvalues of $\hat{\boldsymbol{g}}$ are $\hat{\lambda}_1 = 3.92612 + \epsilon(-0.91996)$, $\hat{\lambda}_2 = 1.87034 + \epsilon(0.44710)$, $\hat{\lambda}_3 = 0 + \epsilon(0)$, and the three principal pitches in the $\boldsymbol{\omega}-$basis are given by $h_1^{\boldsymbol{\omega}} = -0.117159$, $h_2^{\boldsymbol{\omega}} = 0.119524$, $h_3^{\boldsymbol{\omega}} = \infty$ respectively. The principal twists, at this configuration, are given by

$$\begin{aligned}
\hat{\boldsymbol{\mathcal{V}}}_1^{\omega} &= (1.61698, -1.11205, 0.27354)^T + \epsilon(0.59693, 1.14580, -0.55239)^T \\
\hat{\boldsymbol{\mathcal{V}}}_2^{\omega} &= (-0.63533, -1.06544, -0.57580)^T + \epsilon(0.13730, -0.28080, -0.02015)^T \\
\hat{\boldsymbol{\mathcal{V}}}_3^{\omega} &= (0, 0, 0)^T + \epsilon(0, 0, 0.90320)^T
\end{aligned}$$

The DOF decoupling is apparent in the purely translational nature of the third principal twist. The principal twists of the $\boldsymbol{h}-$basis are computed as

$$\begin{aligned}
\hat{\boldsymbol{\mathcal{V}}}_1^{h} &= (1.10500, -1.35959, 0)^T + \epsilon(0.55638, 0.85062, -0.80373)^T \\
\hat{\boldsymbol{\mathcal{V}}}_2^{h} &= (9.73597, 7.91281, 6.20082)^T \times 10^{-9} + \epsilon(0, 5.25180, -0.90320 \times 10^9)^T \times 10^{-9} \\
\hat{\boldsymbol{\mathcal{V}}}_3^{h} &= (9.73597, 7.91281, 6.20082)^T \times 10^{-9} + \epsilon(0, 5.25180, 0.90320 \times 10^9)^T \times 10^{-9}
\end{aligned}$$

The principal pitches in $\boldsymbol{h}-$basis are $h_1^h = -0.17648$, $h_2^h = -2.85962 \times 10^7$, $h_3^h = 2.85962 \times 10^7$ respectively. It may be noted that $h_3 = -h_2 \to \infty$, and $\|\hat{\boldsymbol{\mathcal{V}}}_2^h\| = \|\hat{\boldsymbol{\mathcal{V}}}_3^h\| \to 0$, even as $\boldsymbol{g}$ has rank 2. The direction of *pure translation* obtained by subtracting $\hat{\boldsymbol{\mathcal{V}}}_2^h$ from $\hat{\boldsymbol{\mathcal{V}}}_3^h$, $(0,0,0)^T + \epsilon(0, 0, 2 \times 0.9032)$, is consistent with the translational velocity obtained in $\boldsymbol{\omega}-$basis. The advantage of exact analytical computation is also clearly seen from the values of the principal screws in $\boldsymbol{h}-$basis. One can observe that some entries are $\mathcal{O}(1)$ whereas others are $\mathcal{O}(10^{-9})$ and most numerical computations will round them off to 0. If they are rounded off to zero, then we will get two translatory modes which is incorrect.

For the 6-DOF hybrid spatial manipulator, shown in figure 2, the active variables are $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \psi_1, \psi_2, \psi_3)^T$, and the passive variable are given by $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^T$. We choose, the link-lengths as $l_1 = 2l_2 = 4l_3 = 1$, $d = 1/2$, $h = \sqrt{3}/2$, and $s = \sqrt{3}/2$. At a non-singular configuration given by $\boldsymbol{\theta} = (0.2, 0.1, 0.3, -1., -1.2, 1)^T$, and $\boldsymbol{\phi}$ given by $(0.3679, 1.4548, 0.8831)^T$, the dual eigenvalues of $\hat{\boldsymbol{g}}$ are computed as $\hat{\lambda}_1 = 0.03 + \epsilon(0.09)$, $\hat{\lambda}_2 = 2.10 + \epsilon(5.45)$ and $\hat{\lambda}_3 = 1496.45 + \epsilon(1070.41)$. The principal pitches are given by $h_1^{\boldsymbol{\omega}} = 1.37$, $h_2^{\boldsymbol{\omega}} = 1.30$, $h_3^{\boldsymbol{\omega}} = 0.36$, and $h_4^{\boldsymbol{\omega}} = h_5^{\boldsymbol{\omega}} = h_6^{\boldsymbol{\omega}} = \infty$. The last three principal twists signify three translational degrees-of-freedom. We consider the singular configuration where all the three fingers are fully stretched[11], and in this configuration we find that the pure dual principal twists vanish identically, signifying the loss of three translational degrees of freedom. For $\boldsymbol{\theta} = (0.0554, -0.0544, -0.8119, -0.8199, 0, 1.5708)^T$, and $\boldsymbol{\phi} = (-1.3300, -1.3300, 0.7854)^T$, there is a gain of a single DOF, and the gained twist is $(0, 1/3, 0)^T + \epsilon(-1/12, 0, 0)^T$ which indicates angular velocity along Y direction and linear velocity in the $X - Z$ plane. In this example too, the analytical computations yield exact directions and not numerically corrupted values.

6 CONCLUSION

In this paper, we have presented a formal algebraic framework for analysis of multi-DOF rigid body motions which is applied towards analysis of parallel and hybrid manipulators. The development and the results presented in this paper are heavily based on the use and properties of dual numbers, vectors and matrices. The main contributions are a) analytical expressions for principal twists for arbitrary multi-DOF rigid body motions, b) the concept of DOF partitioning, and c) analytical identification of gained or lost twists at singular configurations of a manipulator.

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A Dual Jacobian of Parallel and Hybrid Manipulators

In parallel manipulators, closed loop mechanisms, and hybrid manipulators, in addition to the actuated joints, we have one or more passive joints. A parallel device with m passive variables has m independent constraint equations of the form $\boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{\phi}) = \mathbf{0}$, where $\boldsymbol{\eta}$ is a m-vector, $\boldsymbol{\theta}$, and $\boldsymbol{\phi}$, are n-and m-vectors denoting the actuated and passive joint variables respectively. Differentiating with respect to time and rearranging[12], we get

$$\boldsymbol{J}_{\eta\theta}\dot{\boldsymbol{\theta}} + \boldsymbol{J}_{\eta\phi}\dot{\boldsymbol{\phi}} = \mathbf{0} \tag{15}$$

At a non-singular configuration, $\boldsymbol{J}_{\eta\phi}$ is invertible, and the *passive joint rates* can be obtained as $\dot{\boldsymbol{\phi}} = -\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta}\dot{\boldsymbol{\theta}}$. For parallel and hybrid manipulators the dual Jacobian may be written, after eliminating $\dot{\boldsymbol{\phi}}$, as

$$\hat{\boldsymbol{J}}_{\text{eq}} = (\boldsymbol{J}_{\omega\theta} - \boldsymbol{J}_{\omega\phi}\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta}) + \epsilon(\boldsymbol{J}_{v\theta} - \boldsymbol{J}_{v\phi}\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta}) \tag{16}$$

and its columns may be considered as *equivalent input screws.*