The Freudenstein Equation and Design of Four-link Mechanisms

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Abstract

Ferdinand Freudenstein (1926-2006) is widely acknowledged to be the father of modern kinematics of mechanisms and machines. His Ph. D. thesis in 1954 and subsequent research papers by him and his students have influenced academic and industrial research, teaching and practice related to the analysis and design of mechanisms and machines throughout the world. In this article, we revisit Freudenstein's thesis and the equation named after him. The Freudenstein equation results from an analytical approach towards analysis and design of four-link mechanisms which, along with its variants, are present in a large number of machines used in daily life.

1 Introduction

Mechanisms and machines have been used since ancient times to reduce human effort, and, since the Industrial Revolution, they have entered and impacted almost all aspects of human society. In their most simplistic description, a mechanism is an assemblage of rigid links (or bars) connected by joints which allow relative motion between the connected links. One (input) link of the mechanism is actuated and another (output) link can be made to perform a desired, often intricate, motion. One of the first well-known examples of a mechanism is the Watt's straight-line linkage. This mechanism was designed by James Watt to pull and push the piston-rod in a double acting steam engine he had invented and which is credited to have started the Industrial Revolution (for details about James Watt and his linkages, see a recent article by Deepak and Ananthasuresh[1] and the references contained therein). In modern times, mechanisms are present in a huge variety of gadgets, devices and systems – in bottle cork openers, in bicycles, in garage door opening system, in steering and braking system of a car, in construction equipment for moving dirt and other material, to move control surfaces of aircrafts, in spacecrafts to deploy solar panels and other appendages, in laparoscopic surgical tools, artificial prosthetic knees and other medical devices, to name a few. In the last 50 years mechanisms have been combined with advanced electronics, sensors, control systems and computing technologies, and this marriage has resulted in devices such as robots, micro-electro mechanical systems (MEMS) and other so-called intelligent products.

In the nineteenth century and prior to 1950's, most mechanism analysis and design was done graphically. During the 1950's computers and algorithms for computing were being rapidly developed in USA and elsewhere. Freudenstein was amongst the first person to realize the potential of computers for analysis and design of mechanisms and machines and his analytical approach fitted perfectly well with the rapidly developing computing technologies. In the next section, to put his work in perspective, the key concepts and steps of the then prevalent graphical approach are presented. In the section 3, the analytical approach developed by Freudenstein in his Ph. D. thesis [2] and initial research publications [3, 4] will be presented. Finally, in section 4 a numerical example of the design of a four-link mechanism based on the Freudenstein equation is presented.

2 Before Freudenstein

Consider the four-link mechanism shown in figure 1. It consists of three movable *rigid* bars or links and a *fixed* frame (according to convention, the fixed frame is also counted as a link giving rise to the number *four* in a four-link or a four-bar mechanism). As shown in figure 1, two consecutive links are connected by a rotary joint which allows relative rotation between the links. By using geometry, it can be shown that for known link lengths *a, b, c, d,* and for a known rotation angle at any rotary joint, the four-link mechanism can be fully described – by this we mean that the *position and orientation of all links* in the mechanism can be completely determined and the mechanism can be drawn on a plain sheet of paper. The steps, for the case of the known angle Φ made by link **AB** with the frame, are as follows: with **A** as the centre, draw a circular arc of radius *b*. Mark the point **B** on the circular arc such that the line AB makes the given angle Φ with the fixed frame[1]. With **D** as the centre, draw a circular arc of radius *d*. With point **B** as centre draw a circular arc of radius *c*. The two circular arcs centered at **B** and **D** can intersect at most at two possible points – let they be denoted by **C** and **C'**. The *two possible* four-link mechanisms for the given link lengths and angle Φ are **ABCD** and **ABC'D** as shown in figure 1.

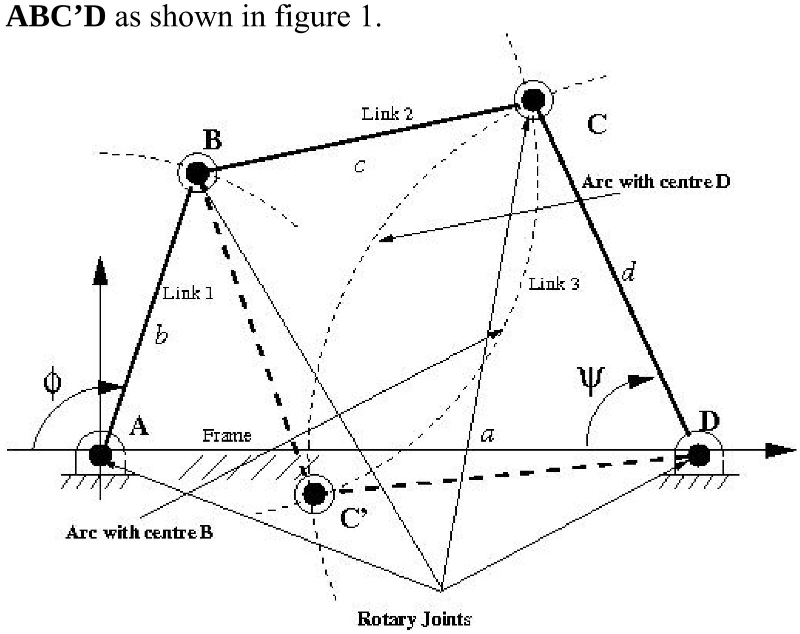

Figure 1: A four-link mechanism showing two possible configurations at a given Φ

[1] We will use the same notation in this article as used by Freudenstein in references [2, 3 and 4].

In the language of kinematics of mechanisms, the four-link mechanism has *one degree of freedom*[2] and it is possible to obtain a complete description of the four-link mechanism when the rotation at any rotary joint is given. Prior to 1954, the prevalent method to obtain the two possible rotations, ψ, of the *output* link **DC**, for a given rotation Φ of *input* link **AB**, was essentially *graphical* and similar to the procedure described above. Freudenstein in 1954 [2] introduced a simple and elegant algebraic expression relating the input angle Φ and the output angle ψ in terms of the link lengths, *a, b, c,* and *d.*

In addition to solving the *analysis* problem, namely obtain values of ψ for a given Φ and link lengths, it is often of engineering importance to *design*[3] a four-link mechanism which can give desired value(s) of ψ for given value(s) of Φ. To design such a mechanism, we need to determine the link lengths and other design variables. Before Freudenstein's work there existed *graphical* approaches for the design of four-link mechanisms where the desired characteristics could be satisfied at a *finite* number of configurations, also called *precision points*, in the range of motion of the mechanism. The error is zero at the precision points and effort was made to minimize the error at other configurations. Typically design was done for three or four precision points. To understand the graphical approach, we review the design of a four-link function generation mechanism for three precision points.

The top half of figure 2 shows three prescribed rotations of input link and the corresponding rotations of the output link. To obtain the four-link mechanism for the given three precision points, we proceed as follows: Fix the output link and rotate the fixed link -- this process is called *inversion* and in inversion the relative motion between various links is not altered. The first movement from $\mathbf{DC_1}$ to $\mathbf{DC_2}$ can be obtained by rotating the fixed link (frame) through $-\psi^{12}$ about point **D**. This results in the point **A** going to $\mathbf{A^2}$. The inverted position of $\mathbf{B_2}$, denoted by $\mathbf{B_2^1}$, can be obtained by constructing the angle $\Phi_s + \Phi^{12}$ from the point $\mathbf{A^2}$ where Φ_s is an unspecified arbitrary starting Φ. Likewise the movement from $\mathbf{DC_1}$ to $\mathbf{DC_3}$ can be obtained by rotating fixed link through $-\psi^{13}$ and the point **A** goes to $\mathbf{A^3}$**.** The inverted position of $\mathbf{B_3}$, denoted by $\mathbf{B_3^1}$, is obtained by constructing the angle $\Phi_s + \Phi^{13}$ at $\mathbf{A^3}$. The moving point **C** is obtained as the centre of the circle passing through $\mathbf{B_1}$**,** $\mathbf{B_2^1}$ and $\mathbf{B_3^1}$ and the desired four-link mechanism $\mathbf{AB_1CD}$**,** with all the graphical construction steps, is shown in bottom part of figure 2. It can be seen that the graphical construction is fairly simple and reduces to obtaining the centre of

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[2] The degree of freedom of a mechanism can be obtained (except for so-called over-constrained mechanisms) by the well known Grübler-Kutzback criteria. For a planar mechanism, the Grübler-Kutzback criteria is given by the equation $DOF=3(n-j-1) +\Sigma^{j}_{i=1} f_i$, where *DOF* is the degree of freedom, *n* is the number of links including the fixed link, *j* is the number of joints and f_i is the degree of freedom for the ith joint. For more details see the review article by Gogu [5].

[3] This is known as *function generation* in design or synthesis of mechanisms. A mechanism with four or more links can also be designed for *motion generation* and *path generation*. In motion generation a link of the mechanism has to be guided in a prescribed manner. In path generation, the floating or coupler link (link not connected to the fixed frame -- link 2 in the four-link mechanism) has to be guided along a prescribed path. For details about design of mechanisms for different tasks, see [6].

a circle passing through three points. For four precision points, the graphical construction is much more complex (see [6] for details).

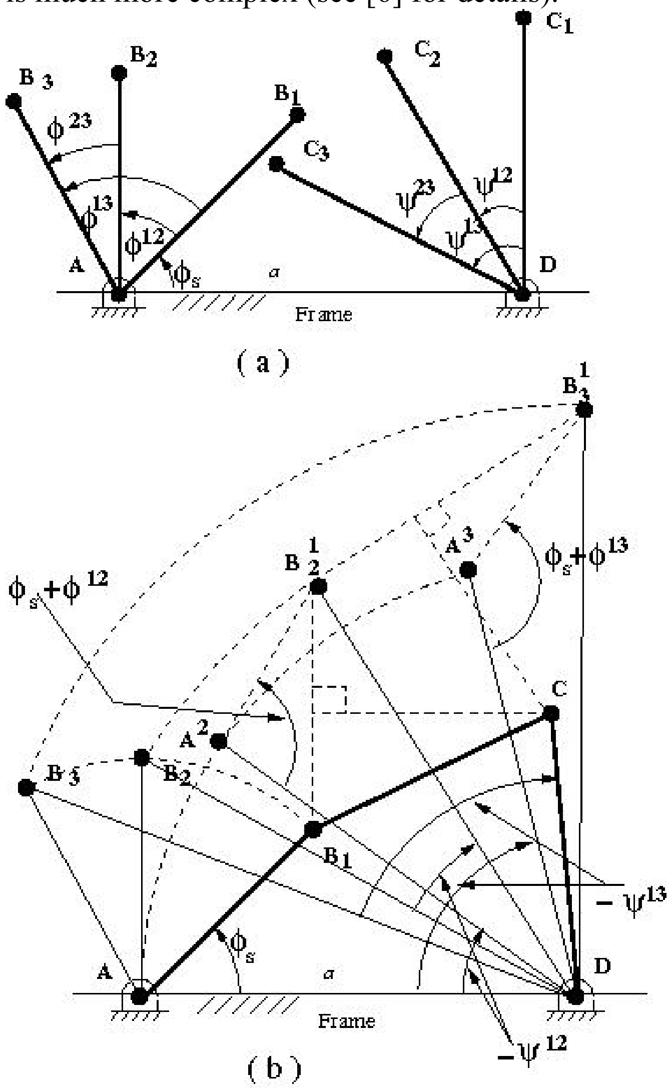

Figure 2: Graphical method for three precision point design of a four-link mechanism

3 Analytical Approach of Freudenstein

In contrast to the graphical approach, Freudenstein developed an analytical approach for analysis and design of four-link mechanisms. In his thesis [2] on page VI-15, he presents an equation (equation 6.19) which relate the rotation angles Φ and ψ in terms of the link lengths *a, b, c* and *d*. This equation also appears in reference [3] and [4] in author's closure and equation (2), respectively. The scalar equation, which is now known as the Freudenstein equation, essentially is the condition for the assembly of the links (also called the loop-closure constraint) in a four-link mechanism at a given Φ. We follow the development of the Freudenstein equation using figure 3 which appears as figure 1 in reference [4].

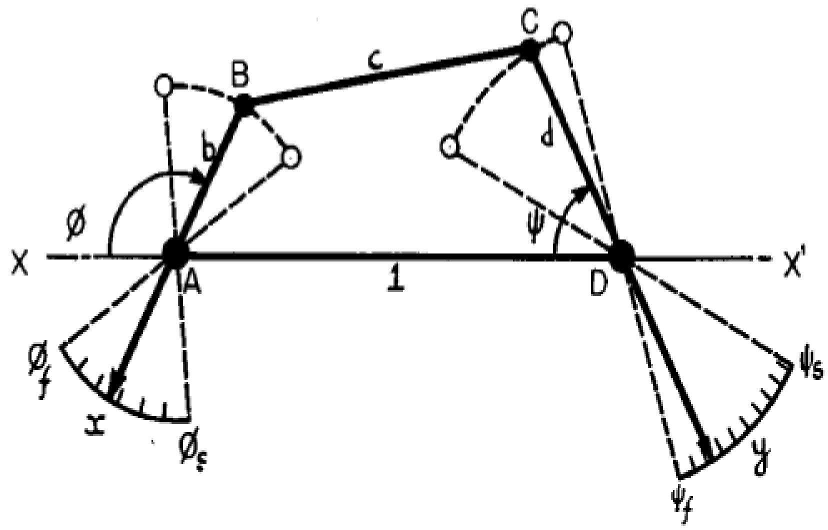

FIG. 1 FOUR-BAR LINKAGE

Figure 3: A four-link mechanism for function generation

In paper [4], the frame is normalized to unity and the other lengths are denoted by *b, c* and *d*, and the input and output angle are Φ and ψ, respectively. The vector **AB** locating point B with respect to A can be obtained in terms of *b* and angle Φ, likewise the vector **AC = AD + DC** can be obtained in terms of 1, *d* and angle ψ. Since the vector equation

AB + BC = AD + DC (1)

must be always satisfied to assemble the four-link mechanism, Freudenstein wrote the scalar equation

$\mathbf{BC} \bullet \mathbf{BC} = (\mathbf{AB} + \mathbf{CD} + \mathbf{DA}) \bullet (\mathbf{AB} + \mathbf{CD} + \mathbf{DA})$ (2)

In the above equation, the vectors **CD** and **DA** are equal to the negative of **DC** and **AD**, respectively, and the symbol `•' represents the vector dot product operation. Simplifying equation (2), Freudenstein obtained a simple scalar equation[4]

$R_1 \cos \Phi - R_2 \cos \psi + R_3 = \cos (\Phi - \psi)$ (3)

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[4] A derivation of equation (3) is as follows: With A as the origin of a X-Y coordinate system, the [*x, y*] coordinates of B and C can be written as [$-b \cos(\Phi)$, $b \sin(\Phi)$] and [$1 - d \cos(\psi)$, $d \sin(\psi)$], respectively. The negative signs arise due to the choice of Φ and ψ by Freudenstein in figure 3. In this derivation, the input and output angles are measured counter-clockwise positive from a horizontal X axis and are thus $\pi - \Phi$ and $\pi - \psi$, respectively. Substituting the [*x, y*] coordinates of B and C, equation (2) becomes $c^2 = [1 - d \cos(\psi) + b \cos(\Phi), d \sin(\psi) - b \sin(\Phi)] \bullet [1 - d \cos(\psi) + b \cos(\Phi), d \sin(\psi) - b \sin(\Phi)]$. On performing the vector dot product, we get $1 + d^2 + b^2 + 2b \cos(\Phi) - 2d \cos(\psi) - 2bd (\cos(\Phi - \psi)) = c^2$. Rearranging and dividing both sides by 2 *bd*, we get equation (3).

where
$R_1 = 1/d$ (4)
$R_2 = 1/b$ (5)
$R_3 = (1 + b^2 - c^2 + d^2)/(2bd)$ (6)

Equation (3) is known as the *Freudenstein Equation* and is readily applicable to kinematics analysis of four-bar mechanisms – from known links lengths and the input angle Φ, the output angle ψ can be found. Using the well known tangent half-angle trigonometric formulas for sine and cosine of angle ψ, it is possible to show that there are two possible ψ's for a given angle Φ – a fact consistent with the graphical results obtained earlier.

Equation (3) can also be directly used for three precision point synthesis for a function generating four-link mechanism. Given three values of input Φ_i, $i=1, 2, 3$, and the corresponding three values of output ψ_i, $i=1, 2, 3$, one can substitute these angle pairs in equation (3) to obtain three *linear* equations in R_1, R_2 and R_3. Once R_1, R_2 and R_3 are obtained from the solution of the linear equations, one can easily obtain the link lengths b, d, and c from equations (4), (5) and (6), respectively. It is interesting to compare the graphical approach and the analytical approach for three precision point design of a four-link mechanism. In the former the centre of a circle is to be determined from three points in a plane whereas in the latter three linear equations need to be solved – both are very straight forward! A second difference is in the choice of the design variables – in the graphical approach the starting rotation of output link $\psi_{\mathbf{s}}$ is determined from the construction (Note: in figure 2, $\mathbf{C_1}$ is not the same as $\mathbf{C}$) where as in the analytical approach this is inherently assumed which in turn yields the three linear equations. Finally, in the analytical approach, the solution of the linear equations may give negative values of R_1 and R_2. Since the link lengths d and b cannot be negative, d and b must be thought of as vectors, and when R_1, R_2 are negative π must be added to the initial angles $\psi_{\mathbf{s}}$, $\Phi_{\mathbf{s}}$, respectively.

For designing with larger number of precision points, Freudenstein introduced two new variables p_i and q_i denoting the rotation angles from unspecified and arbitrary starting positions $\Phi_{\mathbf{s}}$ and $\psi_{\mathbf{s}}$. Setting $\Phi = \Phi_{\mathbf{s}} + p_i$ and $\psi = \psi_{\mathbf{s}} + q_i$, equation (3) now can be written as

$$R_1 \cos(\Phi_{\mathbf{s}} + p_i) - R_2 \cos(\psi_{\mathbf{s}} + q_i) + R_3 = \cos[(\Phi_{\mathbf{s}} + p_i) - (\psi_{\mathbf{s}} + q_i)],\ i = 1, 2, 3, 4, 5 \quad (7)$$

The above equation (7) can be used for four and five precision point synthesis. In his thesis [2] and his paper [4], Freudenstein develops a detailed solution for function generation with four and five-precision point synthesis for a four-link mechanism. Finally, to extend the equation for six and seven precision-point synthesis, Freudenstein introduced scale factors r_Φ and r_ψ to convert the input and output angles into functional variables x and y related by $y = f(x)$, and p_i and q_i were rewritten as $p_i = r_\Phi (x_i - x_s)$ and

$q_i = r_\psi(y_i - y_s)$. Since the scale factors are unspecified, two new variables are added to equation (7) and Freudenstein could achieve up to seven precision point synthesis for a function generating four-link mechanism. It may be noted that for more than three precision points *non-linear transcendental* equations (7) need to be solved – in fact solution of non-linear equations goes hand-in-hand with modern kinematics of mechanisms and machines!

In addition to finite precision point synthesis, Freudenstein also derived detailed formulation to design four-link mechanisms when only one precision point together with a number of derivatives, such as velocity and acceleration, are prescribed. One can clearly see the power and elegance of the Freudenstein analytical approach when four or more precision points are to be used or when derivative information is required to be used – in the graphical approach, the geometry constructions become very complex whereas Freudenstein's approach can be easily programmed in a computer. In the next section, we present a numerical example to illustrate the design of a four-link mechanism for function generation given three precision points.

4 Numerical Example

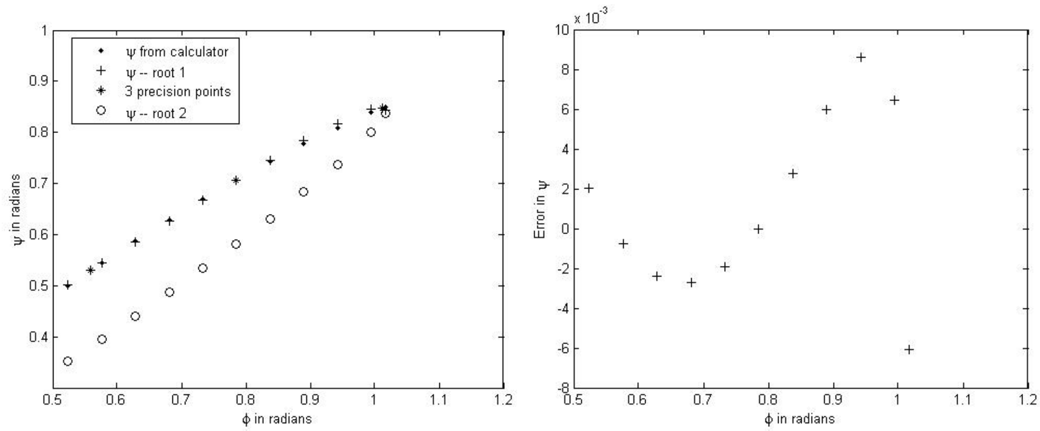
In this example, the aim is to design a four-link mechanism to approximate the function $\psi = \sin(\Phi)$ in the range $\pi/6 < \Phi < \pi/3$ with Φ in radians. We are given 3 precision points, namely, (0.5587, 0.5301), (0.7854, 0.7071) and (1.0121, 0.8479)[5]. Using these sets of values of Φ and ψ in equation (3), we get $R_1 = 0.1890$, $R_2 = 0.2337$, and $R_3 = 1.0410$. From equations (4) through (6), the link lengths for the four-link mechanism are obtained as $a = 1$, $b = 4.2790$, $c = 0.4123$ and $d = 5.2921$. To test how accurate is our design, we plot the values of ψ from equation (3) versus Φ in the range $\pi/6 < \Phi < \pi/3$ in steps of $\pi/60$ radians. In the same plot, we also plot ψ obtained from an electronic calculator for the same values of Φ. In figure 4(a), the symbol '•' shows values obtained using a calculator, the symbol '+' and 'o' show the two values obtained from the Freudenstein equation (3), and the symbol '*' are the given three precision points. In figure 4(b), we plot the error between the'•' values and the '+' values of ψ as a function of Φ. We can make the following observation from figure 4:

- As mentioned earlier, there are two possible solutions for ψ for a given Φ and the solutions lie on two *branches*. Except at a *singularity,* where the two branches meet*,* the output angle ψ of a four-link mechanism cannot switch from one branch to another. In this numerical example, the 3 precision points lie on a *single branch* (see figure 4 (a)) and in this sense, the design does not suffer from the so-called branch defects. This does not happen always and the reader can verify that for a desired range $0 < \Phi < \pi/2$ and three precision points given by (0.1052, 0.1050), (0.7854, 0.7071) and (1.4656, 0.9945), both the branches of the output angle ψ*,* for the designed mechanism, does not contain all the three precision points.

[5] Obtaining precision points to minimize error over the *full range* of input variable has been a research topic. The values used in this example were obtained by using Chebyshev spacing (see [6] for details).

- The error between the values of output angle ψ obtained from an electronic calculator and Freudenstein's equation (3) is quite small, of the order of 10^{-2}, in the entire range. This may not be so for other desired ranges of the input or for other function generators. For better approximation one can use a four or higher precision point synthesis.

- In this example, the two branches of the output angle are close to meeting near π/3 and hence the four-link mechanism is close to a singularity at the end of the range. This is, however, not by choice.



(a) (b)

Figure 4: Generation of function $\psi = \sin(\Phi)$ using Freudenstein equation

5 Conclusions

Freudenstein's contribution to the theory of mechanisms and machines is now widely recognized and the equation named after him is now present in almost all textbooks on analysis and design of mechanisms. It is interesting to note that when his seminal papers appeared, researchers immediately realized the importance of his work. This can be seen from the discussions accompanying the papers. A. S. Hall Jr. in his commentary writes ``… that the author has succeeded in reducing the analytical approach to the 4-bar problem to as uncomplicated a form as possible…..analytical treatment may not be any more difficult or time-consuming than the older graphical-geometric methods…''[4, page 860]. His continuing influence on the mechanisms and machines community and its offshoots, such as robotics, can be seen from the list of more than 575 names (as of June 2010 and growing!) in the Freudenstein academic family tree available at the website http://my.fit.edu/~pierrel/ff.html. Interested readers are referred to reference [7] for an account of the numerous other contributions of Ferdinand Freudenstein and his students.

Acknowledgement

The author wishes to thank Bernard Roth, Asok K Mallik, and G K Ananthasuresh for their comments and useful leads. The author also thanks the students A. Chandra Sekhar, Midhun Sreekumar and Sangamesh R. Deepak for checking the numerical example.

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