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ANALYSIS OF FINITE SELF MOTION AND FINITE DWELL IN CLOSED-LOOP MECHANISMS AND PARALLEL MANIPULATORS

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ABSTRACT

In this paper, we present the necessary and sufficient criteria for finite self motion and finite dwell of the passive links of a parallel manipulator or a closed-loop mechanism. We study the first order properties of the constraint equations associated with the kinematic constraints inherent in a closed-loop mechanism or a parallel manipulator, and arrive at the criteria for the mechanism to gain a degree-of-freedom at a singular point of its workspace. By analyzing the second order properties of the constraint equations, we show that the gain of degree-of-freedom may lead to finite self motion of the passive links if certain configurational and architectural criteria are met. Special configurations and architecture may also lead to finite dwell of the passive links, and the criteria for the same has been derived. The results are illustrated with the help of several closed-loop mechanisms.

Introduction

Parallel manipulators and closed-loop mechanisms show very complicated and diverse behavior at singularities or singular configurations. It is well known that in serial manipulators, singularities lead to the loss of one or more degree-of-freedom. However, a fully in-parallel device can only gain degree-of-freedom as shown by Hunt *et al.* (Hunt, Samuel and McAree, 1991), a hybrid parallel manipulator may both *lose* or *gain* one or more degree-of-freedom at a singular configuration. This gain or loss has been attributed to the degeneracy of two different Jacobian matrices originating from the time derivative of the input-output equation of the manipulator or closed-loop mechanism(Gosselin and Angeles, 1990; Zlatanov, Fenton and Benhabib, 1995). Several researchers have also studied singularities in terms of statics(see, for example, (Merlet, 1991; Agarwal and Roth, 1992; Dasgupta and Mruthyunjaya, 1998; Choudhury and Ghosal, 2000)), and have made use of the force transformation matrix. They have shown that at singular configurations, a parallel manipulator or a closed-loop mechanism cannot withstand external forces or torques in certain directions. Irrespective of the approach, the problem of finding the general criteria for singular configurations is a difficult one, and a few results exist only for certain restricted classes of mechanisms. For example, Basu and Ghosal(Basu and Ghosal, 1997) have obtained the general singular configuration for platform type closed-loop mechanism containing sphere-sphere links.

While most of the existing work on singularity in a parallel manipulator or closed-loop mechanism focus on identification and classification of singularities arising out of the configuration of the device, there has been some work related to the effect of architecture on singularities (Gosselin and Angeles, 1990). The configuration leading to the gain of degree-of-freedom at a singularity may persist over a finite domain of the workspace of the mechanism and allow the passive joints to move finitely even when all active joints are locked, if certain architectural criteria are met. Similarly, special architecture and configurations may result in a passive link losing motion even when the actuators are moved. The present work deals with these two cases of degener-

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ate motion.

In this paper, we focus on the constraint equations associated with the kinematic constraints inherent in a parallel manipulator or closed-loop mechanism. In these mechanisms, there exist *holonomic constraints* in the form of *loop-closure* equations. A parallel manipulator described by n configuration variables with m loop-closure constraint equations will have only $n-m$ degrees-of-freedom – only $n-m$ of the configuration variables can be actuated and m of them are passive. The motion of the m passive joints is governed by the constraint forces which are *normal* to the configuration manifold, and do not do any work. We relate these forces with the first order properties of the constraint equations, and identify the gain of degree-of-freedom at a singular point with the degeneracy of these forces. This is different from the concept of analyzing the loss and gain of velocities, in the *tangent* space of the configuration manifold, done by most researchers. The main contribution of this paper, obtained as a result of analyzing the second order properties of the constraint equations, is a set of results regarding the *architectural* singularity described by Gosselin and Angeles(Gosselin and Angeles, 1990). In particular, we are able to derive the *necessary* and *sufficient analytical criteria* for finite self-motion and finite dwell of links associated with the passive joints of a closed-loop mechanism or a parallel manipulator.

The paper is organized as follows: In section 2, we present the mathematical approach for the analysis of constraint forces and configuration-space singularities of parallel manipulators and closed-loop mechanisms. In section 3, we present the derivations of the necessary and sufficient criteria for finite self-motion, finite dwell of the passive links. In section 4, we illustrate our theoretical results with the help of planar 4-bar and 5-bar mechanisms, and a three-degree-of-freedom planar multi-loop mechanism. Finally, in section 5, we present the conclusions.

Singularities in the Configuration Space

In this section, we discuss the first order properties of the constraint equations, and relate them with the constraint forces and singularities leading to gain of degree-of-freedom. In general, the forward kinematics of a parallel manipulator or closed-loop mechanism can be expressed as a set of relations

$$\mathbf{X} = \psi(q_1, ..., q_n) \tag{1}$$

where, $\mathbf{X}$[1] represents the output variable and $q_i,\ i = 1, ..., n$ is a set of n joint variables, which determine the configuration of the mechanism completely, and form the *configuration-space* of the mechanism or manipulator. The vector function ψ depends on the chosen output link, the geometry and structure of the manipulator and its dimensions. In the case of parallel manipulators and closed-loop mechanisms, not all the n joint variables are actuated and m of them may be passive. In such a case, the degree of freedom of the parallel manipulator or the closed-loop mechanism is $(n-m)$, and in addition to the above equations, we have m independent constraint equations of the form

$$\eta(q_1, ..., q_n) = \mathbf{0} \tag{2}$$

where $\eta(\cdot) = 0$ denotes the m constraint functions, $\eta_i(\cdot) = 0, i = 1, 2, .., m$.[2] Such constraints arise from the *loop-closure* equations of the parallel mechanisms, and as ψ, they also depend on the architecture and geometry of the mechanism. The set of equations, $\eta(q_1, ..., q_n) = \mathbf{0}$, in general represent m constraint hyper-surfaces in the configuration space on which all configuration variables are forced to lie by the loop-closure equations. Differentiating the m constraint equations (2) with respect to time, t, we get

$$\frac{\partial \eta}{\partial t} + \sum_{i=1}^{m} \frac{\partial \eta}{\partial q_i} \dot{q}_i = \mathbf{0} \tag{3}$$

In closed-loop mechanisms and parallel manipulators, the constraint equations, typically, have no explicit dependence on time, and hence we can write

$$\sum_{i=1}^{m} \frac{\partial \eta}{\partial q_i} \dot{q}_i = \mathbf{0} \tag{4}$$

The above equation may be written in matrix form as

$$[\mathbf{N}]\dot{\mathbf{q}} = \mathbf{0} \tag{5}$$

where $\dot{\mathbf{q}}$ denotes the time derivatives of the configuration space variables, and is given by the vector $(\dot{q}_1, \dot{q}_2, ..., \dot{q}_n)$. We can also write the matrix $[\mathbf{N}]$ as $[\mathbf{N}] = (\mathbf{N}_1^T, \mathbf{N}_2^T, ..., \mathbf{N}_m^T)^T$ where $\mathbf{N}_i$, the i^{th} row vector of $[\mathbf{N}]$, is the *gradient vector* to the i^{th} constraint hyper-surface in the configuration space. This implies that the motion of the system in the configuration space is orthogonal to the *normals* to all the constraint surfaces.

Geometric Description of the Constraint Forces

It is well known that associated with m kinematic constraints, as in equation (2), there exist m constraint forces which do not do any work(see, for example, pp. 218-224 in (Haug,

[1] The output may be position and orientation of a chosen output link.

[2] In this paper, we restrict ourselves to non-redundant manipulators and closed-loop mechanisms, i.e., $(n-m) \leq 3$.

1989)). Denoting the constraint forces by the vector $\mathbf{F}_c$, we can write

$$\mathbf{F}_c^T \dot{\mathbf{q}} = \mathbf{0} \tag{6}$$

By comparing equation (6) with equation (5), we get

$$\mathbf{F}_c = \sum_{i=1}^{m} \lambda_i \mathbf{N}_i = [\mathbf{N}]^T \lambda \tag{7}$$

where λ_i's are the components of a *non-null* $m \times 1$ vector, λ. The significance of λ and its connections with singularities in closed-loop mechanisms and parallel manipulators are discussed next.

Relationship between λ, $[\mathbf{N}]$ and Singularity

Since the constraint forces are completely determined by λ and $[\mathbf{N}]$, it is instructive to consider the situation when the matrix $[\mathbf{N}]$ loses rank. Mathematically, it implies that the rows of the matrix $[\mathbf{N}]$ become linearly dependent at that point, and at least one of the gradient vectors, $\mathbf{N}_i$, may be expressed as a linear combination of the others. This implies that the contribution to $\mathbf{F}_c$ from this constraint gets algebraically added to the contribution from the others. Equivalently, we can say that at that point, one or more kinematic constraints is no longer active. This intuitive notion can be more rigorously explained as follows:
From equation(7), the square of the norm of $\mathbf{F}_c$ is computed as

$$\mathbf{F}_c^T \mathbf{F}_c = \lambda^T [\mathbf{N}][\mathbf{N}]^T \lambda = \lambda^T [\mathbf{g_f}] \lambda \tag{8}$$

where $[\mathbf{g_f}]$ is a symmetric, positive definite matrix of dimension $m \times m$ if $[\mathbf{N}]$ is of full rank. Under a constraint on λ of the form $\lambda^T \lambda = k^2, k \in \Re$, the admissible distribution of $\mathbf{F}_c$ may be shown to be an m-dimensional ellipsoid, whose semi-axis lengths are equal to the square-roots of the eigenvalues of $[\mathbf{g_f}]$, and axis-orientations are given by the corresponding eigenvectors. When $\mathbf{N}$ loses rank, $[\mathbf{g_f}]$ also loses rank, and one or more of the eigenvalues of $[\mathbf{g_f}]$ become(s) zero. Correspondingly, the $\mathbf{F}_c$ distribution loses one or more dimension and we now have an ellipsoid of dimension $(m-1)$ or less. In such degenerate cases, the admissible distribution of $\mathbf{F}_c$ is restricted to the reduced row-space of $[\mathbf{g_f}]$, spanned by the eigenvectors corresponding to the non-zero eigenvalues of $[\mathbf{g_f}]$. Equivalently, we cannot have any constraint force in the null-space of $[\mathbf{g_f}]$. From the well-known duality of forces and velocities, we expect the mechanism to gain a degree-of-freedom due to this *local relaxation of constraints* at a singular point. This means that there exists a velocity in the configuration space, even if the actuators are all locked. We now analyze the *gained velocity* with all the actuators in a locked condition.

Gained Velocity with Actuators Locked

Equation (5) may be decomposed into active and passive parts as

$$[\mathbf{K}]\dot{\theta} + [\mathbf{K}^*]\dot{\phi} = \mathbf{0} \tag{9}$$

where we denote the $(n-m)$ active variables by the vector θ and the m passive variables by the vector ϕ, such that $\mathbf{q} = (\theta^T, \phi^T)^T$, and $\dot{\theta}$, $\dot{\phi}$ are the time-derivatives of θ and ϕ respectively. The columns of $[\mathbf{K}]$ and $[\mathbf{K}^*]$ contain the partial derivatives of η with respect to θ and ϕ respectively. Hence, $[\mathbf{N}]$ is of the form

$$[\mathbf{N}] = [[\mathbf{K}]|[\mathbf{K}^*]] \tag{10}$$

With actuators locked ($\dot{\theta} = \mathbf{0}$), the relevant part of equation(9) is $[\mathbf{K}^*]\dot{\phi} = \mathbf{0}$, and effectively, $[\mathbf{K}^*]$ plays the role of $[\mathbf{N}]$. All the above observations made in terms of $[\mathbf{N}]$ are equally valid for $[\mathbf{K}^*]$, and indeed, the rows of $[\mathbf{K}^*]$ may be visualized as the projections of $\mathbf{N}_i$ onto the passive subspace of the configuration manifold. Hence the gain of degree-of-freedom in the configuration space requires the rows of $[\mathbf{K}^*]$ to become linearly dependent, i.e., *singularity criterion* is given by[3]

$$\det[\mathbf{K}^*] = 0 \tag{11}$$

From equation (9), we can still have a nonzero $\dot{\phi}$ when $\det[\mathbf{K}^*] = \mathbf{0}$. This *gained velocity* in the configuration space is in the null-space of $[\mathbf{K}^*]$, and the total motion in the configuration space is in the null-space of $[\mathbf{N}]$. Since the constraint forces are restricted to the row-space of $[\mathbf{N}]$, the constraint forces are orthogonal to the configurational velocity. In other words, *the loss-space of* $\mathbf{F}_c$ *is the gain space for* $\dot{\phi}$.

Second Order Analysis of Constraint Equations

In the previous section, we have analyzed the matrices, $[\mathbf{N}]$, $[\mathbf{K}]$ and $[\mathbf{K}^*]$ which arise from the first derivative of the constraint equations. In this section, we analyze the second-order properties related to the derivatives of $\det[\mathbf{K}^*]$ and elements of $[\mathbf{N}]$. We show that the second-order analysis leads to analytical criteria for finite self motion and finite dwell.

Gain Singularity and Finite Self Motion

Finite self motion(FSM) of a mechanism refers to finite movement of the passive parts of the mechanism, with the actuators held fixed. This clearly forms a subset of gain-singularity,

[3] It may be noted that $[\mathbf{K}^*]$ is always a $m \times m$ matrix, as there are always m constraint equations and m passive variables ϕ in a $n-m$ degree-of-freedom closed-loop mechanism or parallel manipulator.

with the distinction that the configuration responsible for the gain-singularity is maintained over a *finite span* of motion of the corresponding passive parts. It is known that FSM imposes restrictions on the architecture of the mechanism(Gosselin and Angeles, 1990), and in the following discussion, we show how these architectural requirements may be obtained from the analysis of the constraint equations along with the singularity criteria. We start by making the following observations:

1. At non-singular configurations, with all actuated joints locked, all passive joints are also locked. This follows from equation (9).
2. For gain singularity, with all actuated joints θ locked, *a necessary condition* is that $\det[\mathbf{K}^*]$ must be zero. $\det[\mathbf{K}^*]=0$ implies that at gain singularity, one or more of the passive joints are *not locked*. The number of *unlocked* or *independent* passive joints is equal to the $\mathcal{N}([K^*])$, the nullity of $[\mathbf{K}^*]$. From equation(9), we know that the corresponding passive motion lies in the null-space of $[\mathbf{K}^*]$. For the purpose of analysis, we partition ϕ in two parts, namely ϕ^i of dimension $\mathcal{N}(K^*)$ and ϕ^d of dimension $m-\mathcal{N}([K^*])$, which denote the *independent* and *dependent* parts of ϕ respectively.
3. In addition to the *necessary* condition, $\det[K^*]=0$, for FSM, *two sufficient conditions* are that a) the constraint equations should be *independent* of the unlocked passive joints ϕ^i's, and b) $\det[\mathbf{K}^*]$ should be *independent* of the ϕ^i's. Hence, we have to look at the second-order properties obtained from the first derivatives of elements of $[\mathbf{K}^*]$.

To derive the mathematical conditions for FSM, we express the constraint equation (2) in the form $\eta(\theta,\phi^i,\phi^d(\phi^i))=0$. Noting that the active variable θ is held fixed, the total derivatives of η may be written as

$$\frac{d\eta}{d\phi^i_j}=\sum_{k=1}^{m}\frac{\partial\eta}{\partial\phi_k}\frac{\partial\phi_k}{\partial\phi^i_j} \qquad (12)$$

Note that $\frac{\partial\eta}{\partial\phi_k}$ gives the k^{th} column of $[\mathbf{K}^*]$, and at a non-singular point, we have $\frac{\partial\phi_k}{\partial\phi^i_j}=\delta_{jk}$, where δ_{jk} is 1 for $j=k$ and 0 otherwise, since the passive variables do not have any explicit dependence on each other. It can be also noted that the *total derivative* of η coincides with the *partial* derivative at a non-singular point. At a singular point, ϕ^d is dependent on ϕ^i, and the quantities $\frac{\partial\phi_k}{\partial\phi^i_j}$ are no longer zero for $\phi_k\neq\phi^i_j$, and we have the equation

$$\sum_{k=1}^{m}\frac{\partial\eta}{\partial\phi_k}\frac{\partial\phi_k}{\partial\phi^i_j}=0 \qquad (13)$$

Equation(13) gives $\mathcal{N}([K^*])\times m$ scalar equations in m ϕ's, and $(m-\mathcal{N}([K^*]))\times\mathcal{N}([K^*])$ $\frac{\partial\phi_k}{\partial\phi^i_j}$'s. We also have the following second-order relationship,

$$\frac{d}{d\phi^i_j}(\det[\mathbf{K}^*])=0, \qquad j=1,\mathcal{N}([K^*]) \qquad (14)$$

which gives $\mathcal{N}([K^*])$ individual relationships. Hence we have a total of $\mathcal{N}(K^*)\times(m+1)$ equations. After eliminating $(m-\mathcal{N}([K^*]))\times\mathcal{N}([K^*])$ $\frac{\partial\phi_k}{\partial\phi^i_j}$'s from them, we can get $\mathcal{N}([K^*])+\mathcal{N}^2([K^*])$ equations in the m ϕ's and linkage parameters. Out of these equations, $\mathcal{N}([K^*])$ equations reproduce same number of individual singularity relationships between the elements of ϕ, and the rest $\mathcal{N}^2([K^*])$ yield an equal number of relationships between the linkage parameters of the passive part of the mechanism, which govern the criteria of FSM. Substituting these relationships and the singularity criteria in the original constraint equation(2), we obtain the configurational and architectural constraints on the *active part* of the mechanism which allows FSM. Note : In literature, the criteria for FSM has been stated as the meeting of the branches of forward kinematics as well as inverse kinematics(Gosselin and Angeles, 1990). We note that the meeting of the branches occurs when the matrix $[\mathbf{K}^*]$ is rank deficient, and one or more row(s) of $[\mathbf{K}^*]^{-1}(-[\mathbf{K}])$ is(are) null respectively. However, this simultaneous degeneracy of $[\mathbf{K}^*]$, and $[\mathbf{K}^*]^{-1}(-[\mathbf{K}])$ is only a special case. The above procedure is illustrated in section 4 with the examples of three single and multi-loop planar mechanisms.

Dwell of Passive Links : Instantaneous and Finite

Dwell of a passive link refers to a situation when the link is at rest for *instantaneous* or *finite* motion of the actuators. While instantaneous dwell (ID) is the more frequent of the two and can occur for *general* architecture of the mechanism, finite dwell (FD) requires some constraints on the architecture, as in the case of FSM. In the following discussion, we explain the methodology to derive the criteria for both ID and FD in a mechanism. We observe the following facts about ID and FD:

1. From equation(9), if $[\mathbf{K}^*]$ is non-singular, we have a non-zero vector $\dot{\phi}$ given by

$$\dot{\phi}=[\mathbf{K}^*]^{-\mathbf{1}}(-[\mathbf{K}])\dot{\theta} \qquad (15)$$

   While all of the $\dot{\phi}_i$'s may not be independent of each other, at a *general* configuration, all of them are nonzero.
2. Velocity of the i^{th} passive joint, $\dot{\phi}_i$, may be zero, if it is not *influenced* by any of the actuated joints. This may be explained as follows. Assuming the passive variables ϕ_i's as explicit functions of θ, we can write the following equation.

$$\dot{\phi}_i=\frac{\partial\phi_i}{\partial\theta_j}\dot{\theta}_j \qquad j=1,n-m \qquad (16)$$

From the above, we find that dwell, or loss of mobility of a passive link associated with joint i will occur with arbitrary input $\dot{\theta}$, if we have $\frac{\partial\phi_i}{\partial\theta_j} = 0, \ j = 1, n-m$. Comparing the last two equations, we note that the element (i, j) of $[\mathbf{K}^*]^{-\mathbf{1}}(-[\mathbf{K}])$ may be written as

$$([\mathbf{K}^*]^{-\mathbf{1}}(-[\mathbf{K}]))_{ij} = \frac{\partial\phi_i}{\partial\theta_j} \qquad (17)$$

and $-([\mathbf{K}^*]^{-\mathbf{1}}[\mathbf{K}])_{ij}$ gives the *kinematic influence* of θ_j on ϕ_i. Since $\frac{\partial\phi_i}{\partial\theta_j}, \ j = 1, n-m$ gives the i^{th} row of $[\mathbf{K}^*]^{-\mathbf{1}}(-[\mathbf{K}])$, we arrive at the following *necessary and sufficient* condition for ID:

The link associated with the i^{th} *joint of a mechanism dwells instantaneously with arbitrary input* $\dot{\theta}$*, if the* i^{th} *row of the matrix* $[\mathbf{K}^*]^{-\mathbf{1}}(-[\mathbf{K}])$*, denoted by* $\mathbf{R}_i$*, becomes null.*

3. Finite dwell (FD) implies that the above condition for ID is maintained over a finite span of motion of the input joints, which implies the second-order properties of the constraints, in the form of derivatives of $\mathbf{R}_i$, have to be studied. Hence we have the following criteria for FD.

   (a) The ID criteria is satisfied, i.e.,

$$\mathbf{R}_i = \mathbf{0} \qquad (18)$$

   (b) The constraint equations are independent of θ, over the span of FD, i.e.,

$$\frac{d\eta}{d\theta} = \mathbf{0} \qquad (19)$$

   (c) The ID criteria is maintained over the span of FD, i.e.,

$$\frac{d\mathbf{R}_i}{d\theta} = \mathbf{0} \qquad (20)$$

The equations (18), (19) and (20) give the *necessary and sufficient* criteria for FD. We now investigate the structure of the above equations, and extract the architectural requirements for FD from them. Expanding equation (19) by chain rule of derivatives, we get

$$\frac{d\eta}{d\theta_i} = \frac{\partial\eta}{\partial\theta_i} + \sum_{k=1}^{m} \frac{\partial\eta}{\partial\phi_k}\frac{\partial\phi_k}{\partial\theta_i} = 0 \qquad i = 1, n-m \qquad (21)$$

The last equation gives a set of $m \times (n-m)$ scalar equations in an equal number of partial derivatives of the form $\frac{\partial\phi_k}{\partial\theta_i}$. Similarly,

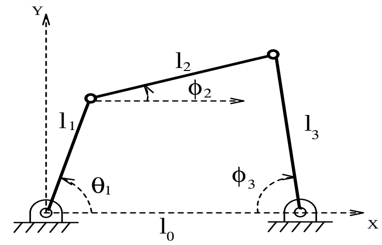

Figure 1. PLANAR 4-BAR MECHANISM

if we have n_d numbers of passive links in finite dwell, we obtain $n_d \times (n-m)^2$ scalar equations in $\frac{\partial\phi_k}{\partial\theta_i}$'s from equation (20). Combining them, we obtain a set of $(n-m) \times (m + n_d \times (n-m))$ equations in $(m - n_d) \times (n-m)$ unknowns[4], $\frac{\partial\phi_k}{\partial\theta_i}$'s. Hence we can eliminate all the partial derivatives from the above *over-constrained* equations, and obtain $n_d \times (n-m) \times (n-m+1)$ *homogeneous* equations in the *architectural and configurational* parameters of the mechanism. These equations yield $n_d \times (n-m)$ conditions giving the ID criteria, and the rest N of them give an equal number of relationships between the architectural parameters involved in the *mobile* part of the mechanism, where N is given by

$$N = n_d \times (n-m)^2 \qquad (22)$$

Substituting relationships obtained above in the original constraint equation(2), we obtain the configurational and architectural constraints on the dwelling part of the mechanism.
Note : The quantity N gives an upper bound on the number of architectural constraints that can be extracted using the above procedure. The actual number, however, will depend on the structure of the constraint equations, hence may be less than N (see examples in section 4).

The above theoretical development is illustrated in the next section with the help of several planar mechanisms.

Illustrative Examples

In this section, we illustrate the theory developed in section 3 with the examples of three planar closed-loop mechanisms.

Planar 4-Bar Mechanism

Figure 1 shows the geometry of a general 4-bar mechanism. The first link is the driving link, and its position is given by the active variable, θ_1, while $\phi = (\phi_2, \phi_3)^T$ gives the passive variable respectively. Hence we have $\mathbf{q} = (\theta_1, \phi_2, \phi_3)^T$. The loop-closure

[4]The number of unknown partial derivatives is reduced by $n_d \times (n-m)$, since these many partial derivatives will be identically zero.

equations for the 4-bar mechanism are given by [5]

$$\begin{aligned}\eta_1 &= l_1c_1 + l_2c_2 + l_3c_3 - l_0 = 0\\ \eta_2 &= l_1s_1 + l_2s_2 - l_3s_3 = 0\end{aligned} \tag{23}$$

The matrices $[\mathbf{K}]$ and $[\mathbf{K}^*]$ are computed from equation (23) as

$$\begin{aligned}[\mathbf{K}] &= \begin{pmatrix} -l_1s_1 \\ l_1c_1 \end{pmatrix}\\ [\mathbf{K}^*] &= \begin{pmatrix} -l_2s_2 & -l_3s_3 \\ l_2c_2 & -l_3c_3 \end{pmatrix}\end{aligned} \tag{24}$$

From equation (11), the condition for *gain* singularity is given by

$$l_2l_3\sin(\phi_2+\phi_3) = 0 \tag{25}$$

Geometrically, this means that the two passive links of length l_2 and l_3 get aligned.

Finite Self Motion of 4-Bar Mechanism It is known that the 4-bar mechanism can show FSM, if the following conditions are satisfied (Gosselin and Angeles, 1990):

1. Architectural requirement : $l_2 = l_3, l_1 = l_0$
2. Configurational requirement : $\theta_1 = 0$

We now derive the above conditions, following the method described in section 3. In this case, $\mathcal{N}([\mathbf{K}^*]) = 1$, we have only one *independent* passive variable, which we choose to be ϕ_2. According to the notations introduced in section 3, we have $\phi^i = \phi_2, \phi^d = \phi_3$. Substituting $\eta = (\eta_1, \eta_2)^T$ from equation (23) into (12), we get the following equations:

$$\begin{aligned}-l_2s_2 - l_3s_3\frac{\partial\phi_3}{\partial\phi_2} &= 0\\ l_2c_2 - l_3c_3\frac{\partial\phi_3}{\partial\phi_2} &= 0\end{aligned} \tag{26}$$

Applying equation (14) to the singularity criteria given by equation (25), we get $\frac{\partial\phi_3}{\partial\phi_2} = -1$. Substituting for $\frac{\partial\phi_3}{\partial\phi_2}$ into equation (26), we get

$$\begin{aligned}-l_2s_2 &= -l_3s_3\\ l_2c_2 &= l_3c_3\end{aligned} \tag{27}$$

Squaring and adding the above last equations, we can eliminate ϕ_2, ϕ_3 to obtain the *architectural requirement* on the passive part

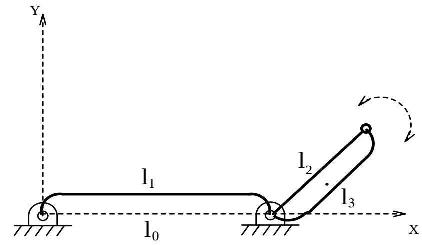

Figure 2. CONFIGURATION OF THE 4-BAR MECHANISM FOR FSM

for FSM as $l_2 = l_3$[6]. Note that the equation (27) also yield the singularity criteria, if we consider them as homogeneous equations in the link-lengths l_2 and l_3, and apply the consistency criterion to these equations. The above results verify our claim regarding the number of architectural constraints when $\mathcal{N}([\mathbf{K}^*])$ is 1.

The architectural and configurational requirement on the *active* part of the mechanism may also be deduced as follows. Substituting the equation (27) into the original constraint equations (23), we get $l_1s_1 = 0$ and $l_1c_1 - l_0 = 0$, where from we obtain the only physically meaningful solution as $\theta_1 = 0$, and $l_1 = l_0$. These are the configurational and architectural requirements on the *active* part of the mechanism respectively. The configuration of the mechanism at this point is shown in figure 2.

Finite Dwell of 4-Bar Mechanism The output link of a 4-bar mechanism can show FD, if the following conditions are satisfied(Gosselin and Angeles, 1990).

1. Architectural requirement : $l_1 = l_2, l_3 = l_0$
2. Configurational requirement : $\phi_3 = 0$

These relations are derived in the following discussion. In this case, we have $n_d = 1$, as the only link in dwell is the output link. From equations (24), we get the matrix $[\mathbf{K}^*]^{-1}(-[\mathbf{K}])$ as

$$[\mathbf{K}^*]^{-1}(-[\mathbf{K}]) = \frac{1}{l_2l_3\sin(\phi_2+\phi_3)}\begin{pmatrix} l_1l_3\sin(\theta_1+\phi_3) \\ l_1l_2\sin(\theta_1-\phi_2) \end{pmatrix} \tag{28}$$

Applying equation (18) to the above equation, we find the criteria for the ID of the third link as

$$\sin(\theta_1 - \phi_2) = 0 \tag{29}$$

assuming the configuration to be non-singular, i.e., $\sin(\phi_2 + \phi_3) \neq 0$. To find conditions for FD we begin by applying equation (19) to equation (23). We get

$$-l_1s_1 - l_2s_2\frac{\partial\phi_2}{\partial\theta_1} - l_3s_3\frac{\partial\phi_3}{\partial\theta_1} = 0$$

[5] c_i, s_i indicate $\cos(q_i), \sin(q_i)$ respectively in this paper.

[6] The positive roots are considered, since the lengths are positive quantities.

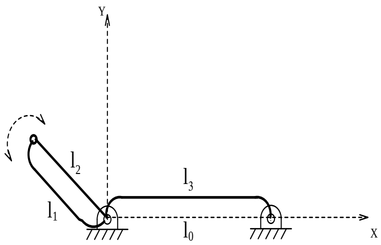

Figure 3. CONFIGURATION OF THE 4-BAR MECHANISM FOR FINITE DWELL

$$l_1c_1 + l_2c_2\frac{\partial\phi_2}{\partial\theta_1} - l_3c_3\frac{\partial\phi_3}{\partial\theta_1} = 0 \tag{30}$$

Applying equation (20) to equation (29), we get $\frac{\partial\phi_2}{\partial\theta_1} = 1$ Substituting for $\frac{\partial\phi_2}{\partial\theta_1}$ into equation (30), and noting that $\frac{\partial\phi_3}{\partial\theta_1} = 0$ by the definition of FD, we get

$$\begin{aligned} -l_1s_1 &= l_2s_2 \\ l_1c_1 &= -l_2c_2 \end{aligned} \tag{31}$$

Squaring and adding these two equations, we can eliminate θ_1, ϕ_2 to obtain one of the *architectural requirements* as $l_1 = l_2$. Note that the equations (31) also yield the ID criteria, if we consider them as homogeneous equations in the link-lengths l_1 and l_2, and apply the consistency criterion to these equations. The above results verify our claim regarding the maximum number of architectural constraints when n_d is 1 (see section 3).

The architectural and configurational requirement on the dwelling part of the mechanism may also be deduced as follows. Substituting the equations (31) into the original constraint equations (23), we get $l_3s_3 = 0$ and $l_3c_3 - l_0 = 0$. The physically meaningful solution of these equations is $\phi_3 = 0$, and $l_3 = l_0$, which are the configurational and architectural requirements on the *dwelling* part of the mechanism respectively. The configuration of the mechanism at this point is shown in figure 3.

Planar 5-Bar Mechanism

Figure 4 shows the geometry of a planar 5-bar mechanism. Link 1 and link 4 are the actuated links, and the *active variable* is given by $\theta = (\theta_1, \theta_4)^T$, while passive variable are given by $\phi = (\phi_2, \phi_3)^T$. Hence we have $\mathbf{q} = (\theta_1, \phi_2, \phi_3, \theta_4)^T$. Loop-closure equations for the 5-bar mechanism is

$$\begin{aligned} \eta_1 &= l_1c_1 + l_2c_2 + l_3c_3 + l_4c_4 - l_0 = 0 \\ \eta_2 &= l_1s_1 + l_2s_2 - l_3s_3 - l_4s_4 = 0 \end{aligned} \tag{32}$$

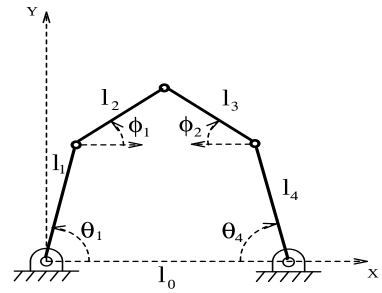

Figure 4. PLANAR 5-BAR MECHANISM WITH REVOLUTE JOINTS

The matrices $[\mathbf{K}]$ and $[\mathbf{K}^*]$ are computed as

$$\begin{aligned} [\mathbf{K}] &= \begin{pmatrix} -l_1s_1 & -l_4s_4 \\ l_1c_1 & -l_4c_4 \end{pmatrix} \\ [\mathbf{K}^*] &= \begin{pmatrix} -l_2s_2 & -l_3s_3 \\ l_2c_2 & -l_3c_3 \end{pmatrix} \end{aligned} \tag{33}$$

The condition for gain-singularity is same as the previous case, i.e., $l_2l_3\sin(\phi_2 + \phi_3) = 0$, implying that the passive links get aligned.

Finite Self Motion of 5-Bar Mechanism We note that the *instantaneous kinematics* of the passive part of a mechanism is determined by the matrix $[\mathbf{K}^*]$, and since $[\mathbf{K}^*]$ is the same for both the 4-bar mechanism and the 5-bar mechanism, the architectural requirement for the passive part of the mechanism is $l_2 = l_3$, i.e., same as that in the case of the 4-bar mechanism. Using this result along with equation (27), we find out the requirements on the active parts of the mechanism. Substituting equations (27) into the original constraint equations (32), we get

$$\begin{aligned} l_1s_1 - l_4s_4 &= 0 \\ l_1c_1 + l_4c_4 - l_0 &= 0 \end{aligned} \tag{34}$$

Eliminating θ_1 from the above equations, we get after some rearrangement

$$c_4 = \frac{l_0^2 + l_4^2 - l_1^2}{2l_0l_4} \tag{35}$$

The above equation implies that the links $1, 0$ and 4 constitute a triangle, in which θ_4 is the angle contained by the links 0 and 4. The corresponding configuration is shown in figure 5.

Planar 3-degree-of-freedom Parallel Manipulator with Revolute Actuators

In this section, we analyze a planar 3-loop, 3-degree-of-freedom manipulator discussed in(Gosselin and Angeles, 1990).

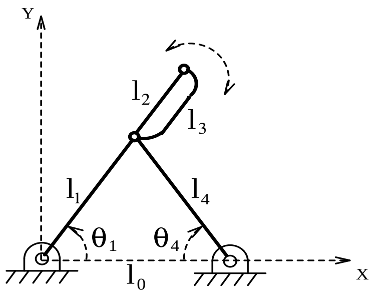

Figure 5. CONFIGURATION OF THE 5-BAR MECHANISM FOR FSM

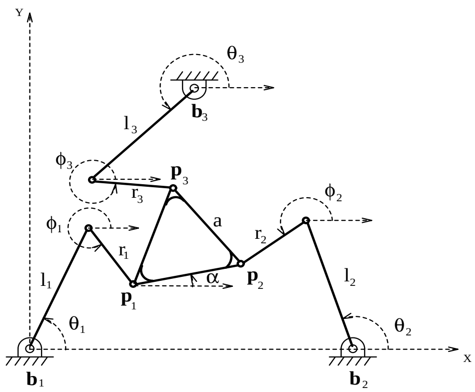

Figure 6. PLANAR 3-DEGREE-OF-FREEDOM PARALLEL MANIPULATOR

Figure 6 shows the geometry of the planar 3-degree-of-freedom manipulator. The pivots are located at the vertices of an equilateral triangle, called the *base* triangle. All the active links, l_i's are connected to the respective pivots, where the motors are located. The gripped object is modeled as an equilateral triangular platform, and the contacts between the tips of the fingers and the object are modeled as *no-slip* contacts. Hence the connection between the tip of a finger and the platform is kinematically equivalent to a rotary joint.

The active variables in this case are given by $\theta = (\theta_1, \theta_2, \theta_3)$, and the passive variables by $\phi = (\phi_1, \phi_2, \phi_3, \alpha)^T$, where θ_i, ϕ_i's have been shown in the figure 6. Note that the orientation of the platform, denoted by α, has also been included in ϕ, and we require 4 independent equations to solve for these 4 quantities.

We can write the loop-closure constraint equations explicitly in terms of the configurational variables, and architectural parameters as

$$\begin{aligned}
\eta_1 &= l_1c_1 + r_1c_{\phi_1} + ac_\alpha - r_2c_{\phi_2} - l_2c_2 - x_2 = 0\\
\eta_2 &= l_1s_1 + r_1s_{\phi_1} + as_\alpha - r_2s_{\phi_2} - l_2s_2 = 0\\
\eta_3 &= x_2 + l_2c_2 + r_2c_{\phi_2} + ac_{(\frac{2\pi}{3}+\alpha)} - r_3c_{\phi_3} - l_3c_3 - x_3 = 0\\
\eta_4 &= l_2s_2 + r_2s_{\phi_2} + as_{(\frac{2\pi}{3}+\alpha)} - r_3s_{\phi_3} - l_3s_3 - y_3 = 0
\end{aligned} \tag{36}$$

where (x_i, y_i) give the coordinates of the i^{th} pivot. The matrices $[\mathbf{K}]$ and $[\mathbf{K}^*]$ are given as

$$[\mathbf{K}] = \begin{pmatrix} -l_1s_1 & l_2s_2 & 0\\ l_1c_1 & -l_2c_2 & 0\\ 0 & -l_2s_2 & l_3s_3\\ 0 & l_2c_2 & -l_3c_3 \end{pmatrix}$$
$$[\mathbf{K}^*] = \begin{pmatrix} -r_1s_{\phi_1} & r_2s_{\phi_2} & 0 & -as_\alpha\\ r_1c_{\phi_1} & -r_2c_{\phi_2} & 0 & ac_\alpha\\ 0 & -r_2s_{\phi_2} & r_3s_{\phi_3} & -as_{(\frac{2\pi}{3}+\alpha)}\\ 0 & r_2c_{\phi_2} & -r_3c_{\phi_3} & ac_{(\frac{2\pi}{3}+\alpha)} \end{pmatrix} \tag{37}$$

From equation (11), the condition for *gain* singularity for the manipulator is given by

$$\sin(\phi_1-\alpha)\sin(\phi_2-\phi_3) + \sin(\phi_1-\phi_2)\sin(\alpha+\frac{2\pi}{3}-\phi_3) = 0 \tag{38}$$

Two classes of singular configuration may be identified from the above expression:

1. All passive links are parallel, and
2. All the passive links, or their hypothetical extensions, intersect at a point.

In both the above cases, $\mathcal{N}([\mathbf{K}^*]) = 1$, and and the null-space of $[\mathbf{K}^*]$ is spanned by a single *non-null* vector $\dot{\phi}$. However, the vector $\dot{\phi}$ has different components in the two cases.

1. If all passive links are parallel, then

$$\begin{aligned} \frac{\dot{\phi}_1}{r_1} + \frac{\dot{\phi}_2}{r_2} + \frac{\dot{\phi}_3}{r_3} &= 0\\ \dot{\alpha} &= 0 \end{aligned} \tag{39}$$

2. If the passive links intersect at a point, the components of $\dot{\phi}_n$ depend on the configuration. In particular, when all the passive links, or their hypothetical extensions intersect at the center of the mobile platform, the components of the gained passive velocity satisfy the equation

$$\frac{\dot{\phi}_1}{r_1} + \frac{\dot{\phi}_2}{r_2} + \frac{\dot{\phi}_3}{r_3} - \sqrt{3}\frac{\dot{\alpha}}{a} = 0 \tag{40}$$

Finite Self Motion Note that for the first case of gained motion, the angular velocity of the mobile platform, given by $\dot{\alpha}$, is zero. However, in the second case, the mobile platform is allowed to have an angular velocity instantaneously at a singular configuration. In addition, if the architecture satisfies certain requirements, the manipulator shows FSM, as the mobile platform can undergo finite rotations even when all the actuators are held fixed. We derive the condition for such motion in the following discussion.

In this case also, we have $\mathcal{N}([\mathbf{K}^*]) = 1$, and the only *independent* passive variable is α. Hence we have $\phi^i = \alpha$, $\phi^d = (\phi_1, \phi_2, \phi_3)^T$. Equating the total derivative of the constraint equations with respect to α to zero, we get the following equations

$$
\begin{aligned}
-r_1 s_{\phi_1}\frac{\partial\phi_1}{\partial\alpha} + r_2 s_{\phi_2}\frac{\partial\phi_2}{\partial\alpha} - a s_\alpha &= 0\\
r_1 c_{\phi_1}\frac{\partial\phi_1}{\partial\alpha} - r_2 c_{\phi_2}\frac{\partial\phi_2}{\partial\alpha} + a c_\alpha &= 0\\
-r_2 s_{\phi_2}\frac{\partial\phi_2}{\partial\alpha} + r_3 s_{\phi_3}\frac{\partial\phi_3}{\partial\alpha} - a s_{(\alpha+\frac{2\pi}{3})} &= 0\\
r_2 c_{\phi_2}\frac{\partial\phi_2}{\partial\alpha} - r_3 c_{\phi_3}\frac{\partial\phi_3}{\partial\alpha} + a c_{(\alpha+\frac{2\pi}{3})} &= 0
\end{aligned}
\tag{41}
$$

Solving the first two equations simultaneously, we get

$$
\begin{aligned}
\frac{\partial\phi_1}{\partial\alpha} &= \frac{a/\sin(\phi_1-\phi_2)}{r_1/\sin(\phi_2-\alpha)}\\
\frac{\partial\phi_2}{\partial\alpha} &= \frac{a/\sin(\phi_1-\phi_2)}{r_2/\sin(\phi_1-\alpha)}
\end{aligned}
\tag{42}
$$

Similarly, solving the last two equations simultaneously, we get

$$
\begin{aligned}
\frac{\partial\phi_2}{\partial\alpha} &= \frac{a/\sin(\phi_2-\phi_3)}{r_2/\sin(\phi_3-\alpha-\frac{2\pi}{3})}\\
\frac{\partial\phi_3}{\partial\alpha} &= \frac{a/\sin(\phi_2-\phi_3)}{r_3/\sin(\phi_2-\alpha-\frac{2\pi}{3})}
\end{aligned}
\tag{43}
$$

It may be verified that equating the two expressions of $\frac{\partial\phi_2}{\partial\alpha}$, we recover the singularity criterion given by equation (38). Differentiating equation (38) with respect to α, we get

$$
\begin{aligned}
&\cos(\phi_1-\alpha)(\frac{\partial\phi_1}{\partial\alpha}-1)\sin(\phi_2-\phi_3)+\sin(\phi_1-\alpha)\cos(\phi_2-\phi_3)\\
&(\frac{\partial\phi_2}{\partial\alpha}-\frac{\partial\phi_3}{\partial\alpha})+\cos(\phi_1-\phi_2)(\frac{\partial\phi_1}{\partial\alpha}-\frac{\partial\phi_2}{\partial\alpha})\sin(\frac{2\pi}{3}+\alpha-\phi_3)\\
&+\sin(\phi_1-\phi_2)\cos(\frac{2\pi}{3}+\alpha-\phi_3)(1-\frac{\partial\phi_3}{\partial\alpha})=0
\end{aligned}
\tag{44}
$$

It may be seen that the last equation is satisfied by

$$
\frac{\partial\phi_i}{\partial\alpha} = 1,\ i = 1,2,3 \tag{45}
$$

We now check if equation (45) is consistent with equations (42), and (43). Using equation(45) in equations (42) and (43), we get

$$
\frac{a}{\sin(\phi_1-\phi_2)} = \frac{r_1}{\sin(\phi_2-\alpha)} = \frac{r_2}{\sin(\phi_1-\alpha)} \tag{46}
$$

The last equation indicates that the links of length r_1, r_2 and a side of the mobile platform form a triangle, where the angles opposite to the sides r_1, r_2 and a are related to the angles $(\phi_2-\alpha), (\phi_1-\alpha)$ and $(\phi_1-\phi_2)$ respectively (see figure 7). Similarly, from equation(43), we find that the links r_2, r_3 form another triangle with another side of the mobile platform. Geometrically, the above conditions require that the ends of the passive links r_1, r_2, r_3 meet at a point, and in this configuration, the passive links can rotate finitely with the platform about this point at the same rate, even when the actuators are held fixed. This observation verifies equation (45).

We now find out the configuration of the active links for FSM. Using equation(45) in equation (41), and substituting in the original constraint equation(36), we get

$$
\begin{aligned}
l_1c_1 - l_2c_2 - x_2 &= 0\\
l_1s_1 - l_2s_2 &= 0\\
x_2 + l_2c_2 - l_3c_3 - x_3 &= 0\\
l_2s_2 - l_3s_3 - y_3 &= 0
\end{aligned}
\tag{47}
$$

Eliminating θ_1 from the first two of the above equations, we get after some rearrangement,

$$
\cos(\pi-\theta_2) = \frac{l_2^2 + x_2^2 - l_1^2}{2l_2x_2} \tag{48}
$$

which shows that the links l_1, l_2 form a triangle with a side of the base triangle, of which, $\pi-\theta_2$ is the angle opposite to the link l_1. Similarly, from the last two equations of (47), we find that the links l_2, l_3 form a triangle with another side of the base triangle. Combining these two conditions, we find that the tips of the active links also meet at a point, which is also the point of meeting of the ends of the passive links, hence the results are consistent. Note that in this case, the architectural requirement is not unique, and any set of link lengths that will allow the manipulator to get into the special configuration shown in figure 7 can lead to FSM.

Finite Dwell The mobile platform of the 3-degree-of-freedom parallel manipulator can also show finite dwell(Gosselin and Angeles, 1990). The architectural requirements for this is derived below using the method described in section 3.

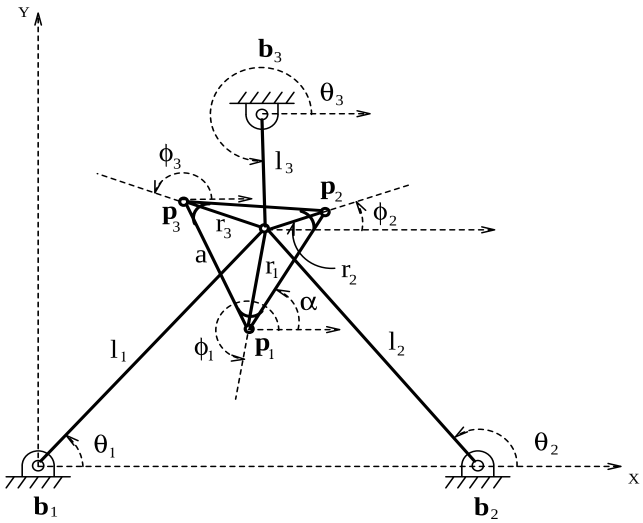

Figure 7. CONFIGURATION OF THE 3-DEGREE-OF-FREEDOM PARALLEL MANIPULATOR FOR FSM

The row of the matrix $[\mathbf{K}^*]^{-1}(-[\mathbf{K}])$ corresponding to $\dot{\alpha}$ is given by

$$[\mathbf{K}^*]^{-1}(-[\mathbf{K}]) = \frac{r_1 r_2 r_3}{\det[\mathbf{K}^*]} \begin{pmatrix} l_1 \sin(\phi_1-\theta_1)\sin(\phi_2-\phi_3) \\ l_2 \sin(\phi_2-\theta_2)\sin(\phi_3-\phi_1) \\ l_3 \sin(\phi_3-\theta_3)\sin(\phi_1-\phi_2) \end{pmatrix}^T \quad (49)$$

We assume $\det[\mathbf{K}^*]$ to be non-zero, hence from equation (38), we must have the ϕ_i's all different. This gives the following criteria for ID: $\sin(\phi_i-\theta_i)=0,\ i=1,2,3$. Differentiating these equations with respect to the active variables θ_j's, and solving for the partial derivatives $\frac{\partial\phi_i}{\partial\theta_j}$, we get $\frac{\partial\phi_i}{\partial\theta_j}=\delta_{ij}$. Differentiating the constraint equations with respect to θ_1, and noting $\frac{\partial\phi_i}{\partial\theta_j}=\delta_{ij}$, we obtain $-l_1 s_1 - r_1 s_{\phi_1}=0$ and $l_1 c_1 + r_1 c_{\phi_1}=0$, where from we recover the ID criterion for this finger of the manipulator as $\sin(\phi_1-\theta_1)=0$, and the architectural requirement, $l_1=r_1$. Similarly, we can obtain $l_2=r_2,\ l_3=r_3$. Thus the architectural constraint on the links of the i^{th} finger is $l_i=r_i$. Note that there is no relationship between the link-lengths of different fingers. Also note that maximum number of architectural requirements on the *non-dwelling* part of the manipulator, as predicted by equation (22) is $1\times(7-4)^2=9$. In this case, however, the actual number is only 3.

To find the constraint on the platform size a, we substitute $l_1=r_1$, and $\phi_1=\theta_1+\pi$ in the constraint equation(36) and obtain $ac_\alpha - x_2 = 0$ and $as_\alpha=0$. From these equations, we find the orientation and the size of the platform as $\alpha=0$ and $a=x_2$. Geometrically, it means that the mobile platform has the same size as the base platform, and rests on top of it for finite dwell. All the passive links fold back on the corresponding active links, such that their tips coincide with the pivots.

Conclusion

In this paper, we have presented an analysis of the constraint equations governing the motion of the passive or non-actuated part of a closed-loop mechanism or parallel manipulator, and shown that the singularities leading to the gain of degree(s)-of-freedom are associated with the degeneracy of these forces. The first order properties govern the instantaneous behavior of the mechanism at a singularity, while the second order properties yield criteria for singular motion over finite span of the configuration space. Second order properties also help obtaining the architectural requirement of finite self-motion at a singularity and finite dwell. The the theory developed in this paper has been illustrated with the examples of three representative closed-loop mechanisms and parallel manipulators.

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