Design of dynamically isotropic two radii Gough-Stewart platforms with arbitrary number of struts

Yogesh Pratap Singh
Dept. of Mechanical Engineering
Indian Institute of Science
Bangalore, India
yogeshsingh@iisc.ac.in

Nazeer Ahmad
Structures Group, URSC
Indian Space Research Organization
Bangalore, India
nazeer@ursc.gov.in

Ashitava Ghosal
Dept. of Mechanical Engineering
Indian Institute of Science
Bangalore, India
asitava@iisc.ac.in

***Abstract* — A class of dynamically isotropic two radii Gough-Stewart Platform (GSP) is considered in this work for the application of isolating micro-vibration. Such a device can attenuate the first six modes of vibration equally and effectively. A novel geometry-based approach employing the force transformation matrix is used to arrive at a complete set of closed-form analytical solutions for any given payload. The above method is applicable to any number of struts (even and odd) for a two radii GSP. A 6-6 dynamically isotropic two radii GSP was found most suitable for the micro-vibration isolation applications compared to other configurations with more struts as it has the lowest natural frequency. The generalized approach presented in this work is based on geometry and has not been discussed in any existing literature. The dynamically isotropic designs obtained from the closed-form solutions were successfully validated using the finite element software ANSYS®.**

Keywords— Dynamic isotropy, Force transformation matrix, Natural frequency matrix, Modified Gough-Stewart Platform (MGSP).

I. INTRODUCTION

Micro-vibration in spacecraft is induced due to the presence of various rotating and reciprocation components such as momentum wheels, reaction wheels, cryo-coolers, and firing of thrusters. A Gough-Stewart platform-based isolator has been extensively used and proposed in the literature [1] for control of micro-vibration. However, these are hard to realize, keeping in mind the non-linearity and time variance associated with the system [2]. Hence several researchers have come up with decoupled and isotropic measures as the desired performance [2-8]. The primary design consideration for effective vibration isolation is that the first six natural frequencies of translation and rotation ideally be the same [1]. In such an ideal design, the slope in the region of isolation for any of the degrees of freedom (DOFs) will not be affected by the peak associated with the cross DOFs in the amplitude vs. frequency curve. The dynamically isotropic configuration ensures all the resonance peaks for different DOFs lie in the same place, making it easy to tune dampers for the given frequency bandwidth. From the point of view of active control, a multi-input-multi-output (MIMO) system can be treated as a single-input-single-output (SISO) system, and a decoupled control strategy can be used [3].

Many researchers have concluded that, practically, a traditional 6-6 Gough-Stewart platform (GSP) cannot be dynamically decoupled [1,4,5,7]. Hence, a modified 6-6 two radii GSP (MGSP) was studied and such a two radii GSP can lead to a dynamically isotropic design [2,3,5,6,7]. It differs from the traditional GSP with the anchoring points described on two radii on each platform instead of one radius in a traditional GSP, as shown in Fig. 1 (See also Fig. 7). Yun et al. [3] designed an isotropic 6-6 MGSP for a telescope secondary mirror and successfully demonstrated its effectiveness experimentally using a decoupled control strategy. Tong et al. [2] and a few others [3,5,6,7,8] explored approaches to develop an analytical solution for an MGSP. In all the cases, the obtained solutions are coupled or implicit and it is challenging to arrive at a design. Additionally, most of these studies are restricted to 6-6 MGSP with six struts. A large class of MGSPs with more than six struts remains unexplored. In practical applications, a greater number of struts (>6) are preferred for distributing heavy loads on several actuators or for fault tolerance [8]. Yi et al. [8] developed a two-parameter class of six-strut orthogonal GSP, leading to an isotropic manipulator, and later, this class was extended to include additional struts in GSPs. However, the formulation was not consistent for any number of struts and for odd-even struts.

To the best of our knowledge, a consistent analytical closed-form solution in their explicit form for any number of struts in an MGSP is yet to be established. The novel geometry-based approach presented in this work provides a closed-form solution in their explicit form, making it easier to directly compare various configurations belonging to this class in terms of geometrical parameters, natural frequencies, and feasibilities.

II. NOMENCLATURE

An MGSP is a parallel manipulator with a movable top platform, a fixed base, and struts with a linear actuator in between them, as shown in Fig. 1. In an MGSP, these struts are divided into two sets having a rotational symmetry for their attachment points along the circumference with an equal angular spacing of 360 divided by the number of legs in that set. Let n_1 be the number of struts in the first set ($(A_1 - B_1)$ in Fig. 1). All struts of this set will be of the same length but rotated by $\theta_1 = (2\pi/n_1)$. Similarly, n_2 be the number of struts in the second set ($(A_j - B_j)$ in Fig. 1 with $j = n_1 + 1$). The rotational symmetry in this set will be by $\theta_2 = (2\pi/n_2)$. Hence an MGSP with n numbers of struts ($\mathrm{n} = \mathrm{n}_1 + \mathrm{n}_2$) will be designated as $\langle \mathrm{n}_1, \mathrm{n}_2 \rangle$ throughout the paper. A 6-6 MGSP with six struts will be designated as $\langle 3, 3 \rangle$ as it has three struts in each set with rotational symmetry of θ_1=θ_2=120°. A $\langle 4, 4 \rangle$ MGSP with eight struts,

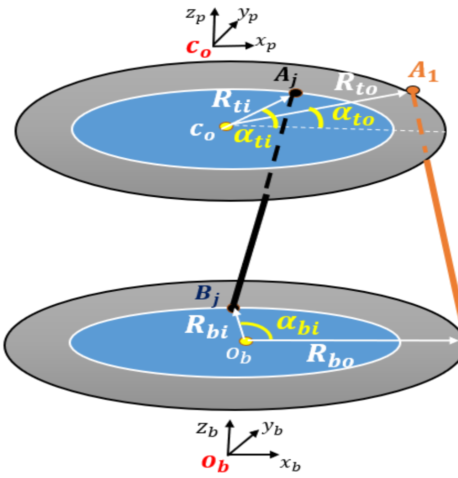

Fig. 1. An MGSP or two radii GSP with one strut belonging to each set.

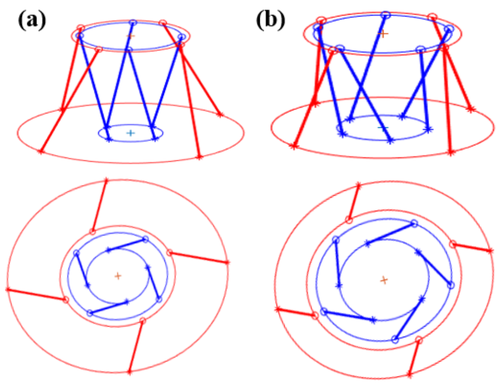

Fig. 2. a) $\langle 4,4\rangle$ MGSP, θ_1=θ_2=90°, b) $\langle 4,5\rangle$ MGSP, θ_1=90°, θ_2=72°.

and $\langle 4,5\rangle$ MGSP with nine struts along with their top views are shown in Fig. 2.

The variables R_{bo}, R_{bi} are used to denote the outer and inner radii of the bottom platform while R_{to}, R_{ti} represents the outer and inner radii of the top platform, respectively in Fig. 1. The coordinates of a point on the base frame $\{B\}$ are given by $\{x_b, y_b, z_b\}$ and on the top or moving frame $\{P\}$ are given by $\{x_p, y_p, z_p\}$. The vector $\boldsymbol{o_b B_1}$ ($i.e.\, R_{bo}$) is chosen along $\boldsymbol{x_b}$, and α_{to}, α_{bi}, α_{ti} are angles made by the anchoring points(A_1, B_j, A_j) of respective radii (R_{to}, R_{bi}, R_{ti}) with $\boldsymbol{x_b}$. The height between the two platforms is denoted by H. The struts are assumed to have a prismatic joint connected to the top platform through a spherical joint and to the bottom platform through a spherical or a universal joint.

III. Formulation

The Jacobian matrix [2-8] or the force transformation matrix [9,10] for GSPs at their neutral position is used and this is a reasonable assumption since in a vibration isolation application, the motion of the top platform is very small. The force transformation matrix ($[\mathbf{B}]$) is a transpose of an inverse Jacobian ($[\mathbf{J}]$) and is given by $[\mathbf{B}] = ([\mathbf{J}]^{-1})^T$[9]. If all the struts are assumed to have an axial stiffness k (joint space), then the stiffness matrix $[\mathbf{K_T}]$ in the task space is described as:

$$[\mathbf{K_T}] = k[\mathbf{B}][\mathbf{B}]^T \tag{1}$$

The force transformation matrix for an MGSP is given by

$$[\mathbf{B}]_{6\times n} = \left[\begin{array}{c|c|c} \boldsymbol{s_1} & \cdots & \boldsymbol{s_n} \\ ({}^B[\mathbf{R}]_P\, {}^P\boldsymbol{p_1}) \times \boldsymbol{s_1} & & ({}^B[\mathbf{R}]_P\, {}^P\boldsymbol{p_n}) \times \boldsymbol{s_n} \end{array}\right] \tag{2}$$

$$\text{where } \boldsymbol{s_q} = \frac{{}^B\boldsymbol{t} + {}^B[\mathbf{R}]_P\, {}^P\boldsymbol{p_q} - {}^B\boldsymbol{b_q}}{l_q}, \text{ where } q\text{=1-}n$$

The vector ${}^B\boldsymbol{t}$ is the vector joining center of top and base platforms, $\boldsymbol{S_q}$ ($= l_q \boldsymbol{s_q}$) is a vector along the respective leg of an MGSP with length l_q, and ${}^P\boldsymbol{p_q}$ and ${}^B\boldsymbol{b_q}$ be the co-ordinates of an anchoring point on the top and base platform expressed in their respective frames. For the neutral position, ${}^B[\mathbf{R}]_P = [\mathbf{I}]$ and ${}^B\boldsymbol{t} = [0\ 0\ H]^T$. An observation in (2) is that moving along the direction of unit vector $\boldsymbol{s_q}$ i.e., $({}^P\boldsymbol{p_q} + \boldsymbol{\delta s_q})$ does not affect the cross product $({}^P\boldsymbol{p_q} + \boldsymbol{\delta s_q}) \times \boldsymbol{s_q} = ({}^P\boldsymbol{p_q}) \times \boldsymbol{s_q}$. Hence given a Jacobian or force transformation matrix, the attachment points on the payloads are not uniquely determined [4,8]. So, we can have an infinite number of configurations with the same $[\mathbf{J}]$ or $[\mathbf{B}]$ matrix.

Let $[\mathbf{M}]$ be the payload's mass matrix in the task space. Without any loss of generality, a diagonal structure of $[\mathbf{M}]$ matrix can be used by choosing the coordinate system to coincide with the orientation of the principal axes of the payload. Hence, $[\mathbf{M}] = \mathrm{diag}([m_p\ m_p\ m_p\ I_{xx}\ I_{yy}\ I_{zz}])$ where, m_p represents the payloads' mass and I_{xx}, I_{yy}, I_{zz} represents its moment of inertia along each direction with respect to its centre of mass (COM).

From $[\mathbf{M}]$ and $[\mathbf{K_T}]$, and using (1), the natural frequency matrix $[\mathbf{G}]$ in the task space [2,4,6,7] is given by

$$[\mathbf{G}] = [\mathbf{M}]^{-1}[\mathbf{K_T}] = [\mathbf{M}]^{-1}k[\mathbf{B}][\mathbf{B}]^T \tag{3}$$

All six eigenvalues of this natural frequency matrix must be equal to obtain dynamic isotropic conditions. Using (2) and (3), the $[\mathbf{G}]$ matrix is given by:

$$[\mathbf{G}] = \begin{vmatrix} \mathbf{P}_{3\times3} & \mathbf{R}_{3\times3} \\ \mathbf{R}^T_{3\times3} & \mathbf{Q}_{3\times3} \end{vmatrix} \tag{4}$$

$$\mathbf{P}_{3\times3} = \mathrm{diag}(\lambda_1,\ \lambda_2,\ \lambda_3)\ ,\ \mathbf{Q}_{3\times3} = \mathrm{diag}(\lambda_4,\ \lambda_5,\ \lambda_6)$$

$$\mathbf{R}_{3\times3} = \begin{bmatrix} \mu_{11} & -\mu_{12} & 0 \\ \mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{bmatrix}$$

$$\lambda_1 = \frac{k\,(n_2 l_1^2 \Psi_1 + n_1 l_2^2 \Psi_2)}{2m_p l_1^2 l_2^2}\ ,\ \lambda_2 = \frac{k\,(n_2 l_1^2 \Psi_1 + n_1 l_2^2 \Psi_2)}{2m_p l_1^2 l_2^2}$$

$$\lambda_3 = \frac{kH^2(n_2 l_1^2 + n_1 l_2^2)}{m_p l_1^2 l_2^2},\ \ \lambda_4 = \frac{kH^2 \Psi_5}{2I_{xx} l_1^2 l_2^2},\ \ \lambda_5 = \lambda_4 \frac{I_{xx}}{I_{yy}}$$

$$\lambda_6 = \frac{k(n_2 \Psi_6^2\, l_1^2 + n_1 \Psi_7^2 l_2^2)}{I_{zz} l_1^2 l_2^2},\ \ \mu_{11} = \frac{-kH(-n_2 \Psi_6 l_1^2 + n_1 \Psi_7 l_2^2)}{2l_1^2 l_2^2},$$

$$\mu_{33} = \frac{kH(-n_2 \Psi_6 l_1^2 + n_1 \Psi_7 l_2^2)}{l_1^2 l_2^2},\ \ \mu_{12} = \frac{kH(n_2 \Psi_3 l_1^2 + n_1 \Psi_4 l_2^2)}{2l_1^2 l_2^2}$$

where,

$$\Psi_1 = R_{ti}^2 + R_{bi}^2 - 2R_{ti}R_{bi}\cos(\alpha_{bi} - \alpha_{ti})$$
$$\Psi_2 = R_{to}^2 + R_{bo}^2 - 2R_{to}R_{bo}\cos(\alpha_{to})$$
$$\Psi_3 = R_{ti}^2 - R_{ti}R_{bi}\cos(\alpha_{bi} - \alpha_{ti})$$
$$\Psi_4 = R_{to}^2 - R_{to}R_{bo}\cos(\alpha_{to})$$
$$\Psi_5 = n_2 R_{ti}^2 l_1^2 + n_1 R_{to}^2 l_2^2$$
$$\Psi_6 = R_{ti}R_{bi}\sin(\alpha_{bi} - \alpha_{ti})$$
$$\Psi_7 = R_{to}R_{bo}\sin(\alpha_{to})$$

As stated, a $\langle n_1, n_2\rangle$ MGSP has two set of struts with lengths:

$$l_1 = |\boldsymbol{S_p}| = \sqrt{R_{to}^2 + R_{bo}^2 - 2R_{to}R_{bo}\cos(\alpha_{to}) + H^2}$$

$$l_2 = |\boldsymbol{S_j}| = \sqrt{R_{ti}^2 + R_{bi}^2 - 2R_{ti}R_{bi}\cos(\alpha_{bi} - \alpha_{ti}) + H^2}$$

Here, $1 \le p \le n_1$, and $n_1 + 1 \le j \le n_1 + n_2$

The relation between these two-leg lengths is given by the leg length ratio denoted by a, and hence,

$$l_2 = a l_1 \tag{5}$$

For dynamic isotropy, we have to solve a set of coupled transcendental equations generated from conditions given by

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \omega^2$$
$$\mu_{11} = \mu_{12} = \mu_{33} = 0.$$

where, ω is the natural frequency of the MGSP and λ_1 to λ_6 are eigen values of matrix $[\mathbf{G}]$.

IV. Design of a Class of MGSP

The difficulty in finding a simple closed-form solution lies in the fact that the number of unknowns is more than the number of transcendental equations. We propose a geometry-based method to obtain the solution.

We start with the condition of $\lambda_1 = \lambda_2 = \lambda_3$, and using (5), we get:

$$b^2 = R_{ti}^2 + R_{bi}^2 - 2R_{ti}R_{bi}\cos(\alpha_{bi} - \alpha_{ti}) \tag{6}$$

$$\text{where, } b = \sqrt{\left(\frac{(n_1(3a^2-1)+2n_2)}{(n_1+n_2)}\right)}\ H \tag{7}$$

Equation (6) can be seen as a cosine rule in a triangle with sides R_{ti}, R_{bi} and b. A similar observation can be made from Fig. 1 where R_{bi} and R_{ti} makes $(\alpha_{bi} - \alpha_{ti})$ angle between them as shown in Fig. 3. Hence the initial variables can be estimated using triangles, Δocq_1, Δocq_2, $\Delta ocq_3 \ldots$ To fix these triangles, the intersection points (i.e., $q_1, q_2, q_{3\ \ldots}$) needs to be determined. For a particular MGSP $\langle n_1, n_2\rangle$ of a given length ratio a and height H, the variable b given by (7) will be a constant. From Fig. 3, $q_1, q_2, q_{3\ \ldots}$ are points equidistant from point c and hence locus is a circle with radius b and center c. If $\{X, Y\}$ is the local coordinate system at o, and $\boldsymbol{X}$ is taken along R_{ti} as shown in Fig. 3, then the equation of virtual circle can be written as:

$$(X - R_{ti})^2 + Y^2 = b^2 \tag{8}$$

Equation of line R_{bi} having a slope $m = \tan(\alpha_{bi} - \alpha_{ti})$ can be written in the same frame as:

$$Y = mX \tag{9}$$

Our geometry-based approach implies that points $q_1, q_2, q_{3\ \ldots}$ can be determined by solving for the intersection of a virtual circle given by (8) and a line R_{bi} at different slopes m given by (9).

For the existence of solution, the condition is given by

$$m = \tan(\alpha_{bi} - \alpha_{ti}) \le \sqrt{b^2/(R_{ti}^2 - b^2)}$$

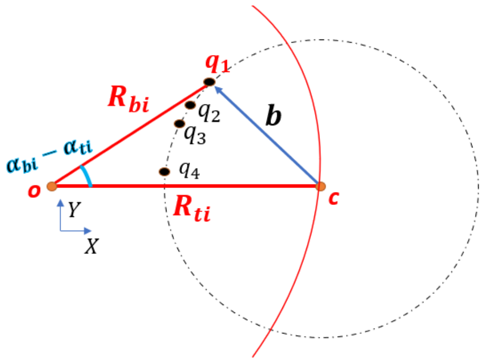

Fig. 3. Top view showing a virtual circle with center $(R_{ti}, 0)$ and radius b intersecting with line R_{bi}.

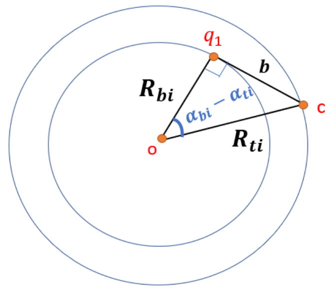

Fig. 4. Top view of an MGSP with a tangency condition.

The maximum value of m at equality will correspond to the tangency condition for the virtual circle, and Δocq_1 will be a right-angle triangle as shown in Fig. 4. We choose this right-angle triangle condition and later generalize it for other intersections apart from the tangency condition. Let,

$$R_{ti} = x\ H \tag{10}$$

where x is a ratio between the two initial parameters and will be found later (Note: $\boldsymbol{X}$ was an axis). Using Pythagoras theorem in Δoq_1c in Fig. 4, we get

$$R_{bi} = \sqrt{(x^2H^2 - b^2)} \tag{11}$$

$$\sin(\alpha_{bi} - \alpha_{ti}) = \frac{b}{x\,H} \tag{12}$$

Let $K = I_{xx}/I_{zz}$ and $Q = I_{xx}/m_p$, hence K and Q (payloads' properties) are known. To satisfying $\lambda_4 = \lambda_5$, $I_{xx} = I_{yy}$ remains a necessary condition and is practically valid for symmetrical payloads. Using the condition $\lambda_4 = \lambda_5 = \lambda_6$, $\mu_{11} = 0$ and (5), on simplification, we get

$$R_{to} = \sqrt{\frac{1}{n_1a^2}\left\{\frac{2n_2^2Kb^2}{H^2a^2}\left(\frac{a^2}{n_2} + \frac{1}{n_1}\right)(x^2H^2 - b^2) - n_2x^2H^2\right\}} \tag{13}$$

Using the condition $\lambda_1 = \lambda_2 = \lambda_3, \mu_{12} = 0,$ and (5), we get

$$R_{bo} = \sqrt{\frac{1}{n_1a^2}\{n_1a^2R_{to}^2 + n_2b^2 + 2H^2(n_1a^2 + n_2)\}} \tag{14}$$

From the condition of $\mu_{11} = 0$, and using (5), we get

$$\sin(\alpha_{to}) = \frac{n_2R_{bi}R_{ti}\sin((\alpha_{bi} - \alpha_{ti}))}{n_1a^2R_{bo}R_{to}} \tag{15}$$

Using the expressions for the variables obtained and the trigonometry identity $\sin^2(\Theta)+\cos^2(\Theta)=1$ in $\mu_{11}=0$, $\mu_{12}=0$, and on simplification, we get

$$x = \sqrt{\frac{K\left(\frac{a^2(2n_1-n_2)+3n_2}{(n_1+n_2)}\right)\left(\frac{(n_1(3a^2-1)+2n_2)}{(n_1+n_2)}\right)^2}{K\left(\frac{a^2(2n_1-n_2)+3n_2}{(n_1+n_2)}\right)\left(\frac{(n_1(3a^2-1)+2n_2)}{(n_1+n_2)}\right)-a^2}} \quad (16)$$

Using $\lambda_3=\lambda_4$ and on simplification, we obtain the expression for H in its explicit form as

$$H = \sqrt{\frac{Qn_1\left(K\left(\frac{a^2(2n_1-n_2)+3n_2}{(n_1+n_2)}\right)\left(\frac{(n_1(3a^2-1)+2n_2)}{(n_1+n_2)}\right)-a^2\right)}{\left\{Kn_2\left(\frac{(n_1(3a^2-1)+2n_2)}{(n_1+n_2)}\right)^2\right\}}} \quad (17)$$

The design procedure for any $\langle n_1,n_2\rangle$ dynamically isotropic MGSP with a given payload (K and Q) can be summarized as:

- Select desired H from (17) with input as ratio a.
- Obtain x from (16) using the same a as above.
- Using (7), b can be obtained with known values of H and a.
- H, a, x, and b can be substituted to find R_{ti}, R_{bi}, $(\alpha_{bi}-\alpha_{ti})$, R_{to}, R_{bo}, α_{to} in the respective order using (10), (11), (12), (13), (14), (15) respectively.

Note that by the above back substitutions, all the parameters can be directly expressed in a more straightforward implicit form with only a as an input, similar to the above steps and is not shown. For example, R_{ti} from (10) can be expressed as:

$$R_{ti} = \frac{Qn_1}{n_2}\left(\frac{a^2(2n_1-n_2)+3n_2}{(n_1+n_2)}\right) \quad (18)$$

V. RESULTS AND OBSERVATIONS

A. *Class of MGSP with $n_1=n_2$*

A very interesting observation that can be made for a class of MGSP with $n_1=n_2$ (equal numbers of struts in both the set $\langle 3,3\rangle,\langle 4,4\rangle,\langle 5,5\rangle$) is that all the geometric parameters remain invariant with n_1 or n_2. It is evident from the closed-form expressions for x, H, and R_{ti} in (16), (17), and (18) which remains the same for this class. Hence an MGSP with $n_1=n_2$ will have all parameters the same except the rotation angle of struts' anchoring points θ_1 and θ_2. A 6-6 MGSP $\langle 3,3\rangle$ ($n_1=n_2$) is also included here.

Fig. 5 (a) and 5 (b) show the plot for different parameters for an MGSP with $n_1=n_2$ for a typical payload. At smaller values of a, R_{bo} is larger making l_1 larger and justifying the smaller value of a (l_2/l_1). Similarly, for a higher value of a, R_{ti} is greater making l_2 larger.

If $n_1 \neq n_2$, the MGSPs will have different values of parameters depending on n_1 and n_2. Fig. 6 (a) and 6 (b) show plot for one such $\langle 3,4\rangle$ MGSP.

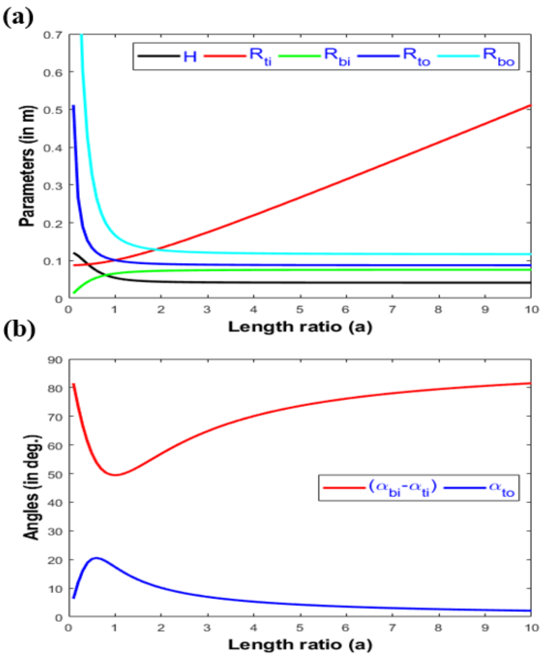

Fig. 5. a) Variation of length parameters for an MGSP where $n_1=n_2$, b) Variation of angles for an MGSP where $n_1=n_2$.

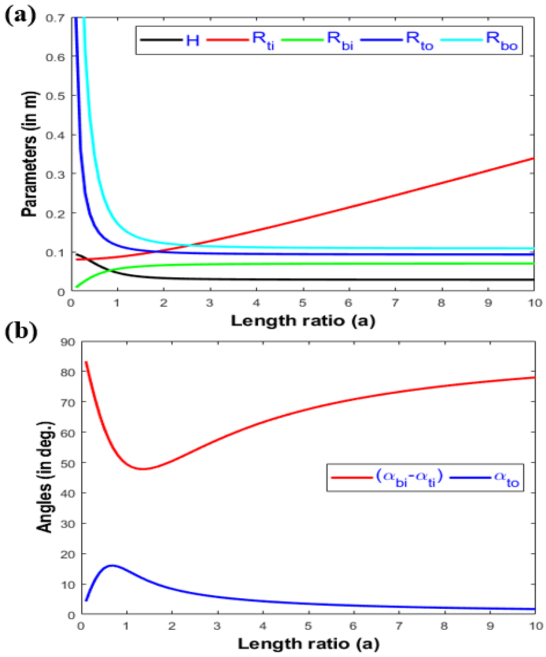

Fig. 6. a) Variation of length parameters for $\langle 3,4\rangle$ MGSP ($n_1\neq n_2$), b) Variation of angles for $\langle 3,4\rangle$ MGSP ($n_1\neq n_2$).

B. *Concept of a transition point (a_*)*

An interesting observation from the Fig. 5 (a) can be seen at a=1, that $R_{to}=R_{ti}$ or the two radii on the top platform converges into a single radius. For the case of MGSP with $n_1=n_2$, this always occurs at a=1, i.e., when the length of both set of struts become equal. Hence, a class of MGSP with all struts equal (a=1), and same number of struts in each set ($n_1=n_2$), will have $R_{to}=R_{ti}$.This points is also a point of configuration transition (a_*) for an MGSP because:

- At $a<(a_*)$, MGSP will have outer to outer radius connections and inner to inner radius connections.

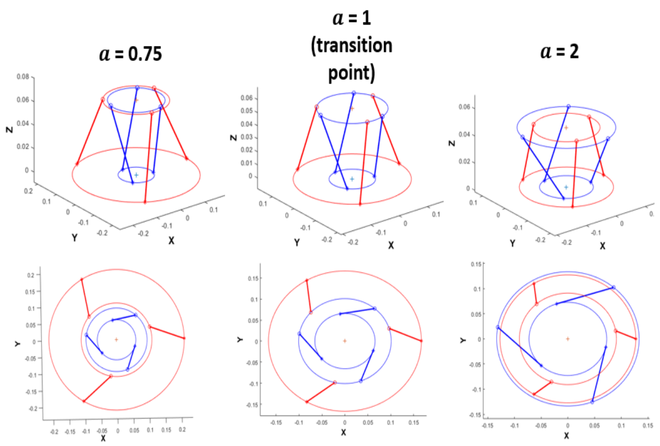

Fig. 7. Configuration transition for $\langle 3,3 \rangle$ MGSP with $n_1 = n_2$ (with their top view).

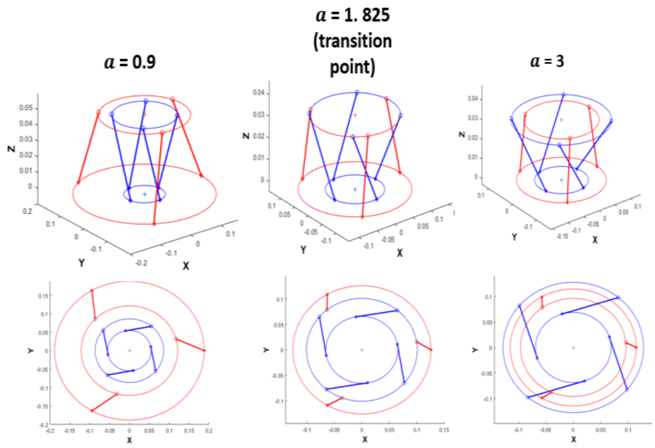

Fig. 8. Configuration transition for $\langle 3,4 \rangle$ MGSP ($n_1 \neq n_2$) (with their top view).

- At $a = (a_*)$, MGSP will have $R_{to} = R_{ti}$.
- At $a > (a_*)$, MGSP will have outer to inner radius connections (cross leg type).

This phenomenon can be visualized from Fig. 7. Using the closed form expression for $R_{to} = R_{ti}$, the expression for (a_*) can be obtained as:

$$a_* = \sqrt{\frac{n_2(2n_2-n_1)}{n_1(2n_1-n_2)}} \tag{19}$$

Interestingly a_* is independent of the payload properties and is always one for MGSP of type $n_1 = n_2$. However, it will hold different values for the case when $n_1 \neq n_2$ which is evident from plot Fig. 6 (a) for $\langle 3,4 \rangle$ MGSP, where $R_{to} = R_{ti}$ is at $a = a_*$=1.825. The phenomenon of configuration transition for a $\langle 3,4 \rangle$ MGSP is given by Fig. 8.

Contrary to this, the value of radius R_{to} or R_{ti} at a_* is dependent only on payload properties. From (18) or (19), the value of this radius is:

$$R_{to} = R_{ti} = \sqrt{2Q}$$

C. General solution for any $\langle n_1, n_2 \rangle$ MGSP

Once all the geometric parameters for an MGSP corresponding to the tangency condition ($a = a_o$) are known, we can fix our virtual circle as shown in Fig. 3 and given by (8). Fixing this circle means keeping the radius b from (7) and offset $(R_{ti}, 0)$ from (18) the same as before $(a = a_o)$. The basic idea is to find all other intersections $(q_1, q_2, q_{3\,...})$ of line R_{bi} with the fixed virtual circle for a general slope m. With R_{ti} and b the same, all other parameters including a are varied to obtain these intersections at any general slope m, which reveals that the condition of tangency is the only solution. This is evident from Fig. 9 where the error $(l_2 - al_1)$ is zero only at single a value $(i.e.\ a_o)$ for which the circle was fixed initially. Three cases for a_o (0.5, 1.0, 2.0) nullify error only at a = 0.5, 1.0, 2.0, respectively. Multiple solution for an MGSP, as discussed before is obtained from the tangency condition for different circles/ different a_o values. Hence, our geometry-based approach effectively provides a complete set of solutions.

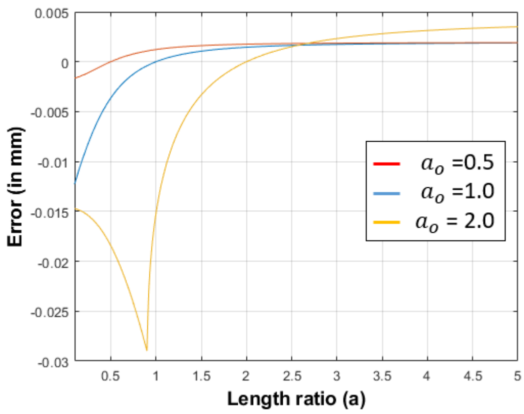

Fig. 9. error $(l_2 - al_1)$ for $\langle 3,3 \rangle$ MGSP at different a_o values.

D. Natural frequencies for any $\langle n_1, n_2 \rangle$ MGSP

For dynamic isotropy, all the natural frequencies ω (square root of the eigenvalues of $[\mathbf{G}]$) must be equal, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \omega^2$ and can be written in closed-form after substituting the values of unknown parameters as

$$\omega = \sqrt{\frac{k(n_1+n_2)}{3m_p}} \tag{20}$$

The translational natural frequencies correspond to λ_1, λ_2, and λ_3 and the rotational natural frequencies correspond to λ_4, λ_5, and λ_6.

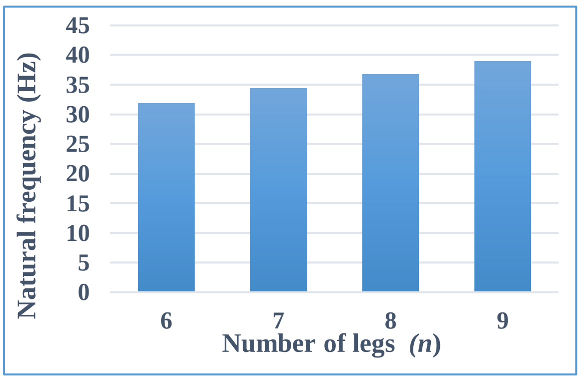

Fig. 10. Natural frequency corresponding to an MGSP.

TABLE I. COMPARISION OF RESULTS OBAINTED USING FE AND CLOSED-FORM.

MGSP Strut Configuration		**Natural frequencies (Hz)**	**DII**
{3,3}	FE	31.678, 31.703, 31.707, 31.810, 31.813, 31.845	1.005
	Closed form	31.83, 31.83, 31.83, 31.83, 31.83, 31.83	1.00
{4,3}	FE	34.218, 34.236, 34.247, 34.359, 34.366, 34.388	1.005
	Closed form	34.38, 34.38, 34.38, 34.38, 34.38, 34.38	1.00
{4,4}	FE	36.543, 36.564, 36.569, 36.723, 36.741, 36.765	1.006
	Closed form	36.75, 36.75, 36.75, 36.75, 36.75, 36.75	1.00

The natural frequency is dependent on the total number of structs $(n_1 + n_2)$. Hence, any $\langle n_1, n_2 \rangle$ dynamically isotropic MGSP with more structs will have a larger value of natural frequency as seen in Fig. 10 for a typical payload with $m_p = 5$ Kg, and $k = 10^5$ N/m. A $\langle 3, 4 \rangle$ MGSP and $\langle 4, 3 \rangle$ MGSP will have the same natural frequencies despite all geometric parameters being different. Interestingly all $\langle n_1, n_2 \rangle$ dynamically isotropic MGSP with $n_1 = n_2$ have a different natural frequency with respect to each other despite having all geometric parameters being the same. The $\langle 4, 4 \rangle$, $\langle 3, 5 \rangle$, $\langle 5, 3 \rangle$ MGSP will have the same natural frequencies pertaining to total eight number of struts.

A 6-6/ $\langle 3, 3 \rangle$ dynamically isotropic MGSP is found to have the least natural frequency among this class and hence, most suitable for vibration isolation purposes. This is because the region of isolation ($\sqrt{2}\omega$) will also appear early in the transmissibility curve, providing more scope for the micro-vibration isolation. Another reason to justify usage of $\langle 3, 3 \rangle$ MGSP over others can be the requirement of a smaller number of actuators. However, we may have to shift to a higher number of struts for heavier payloads, owing to limited load bearing capacities of actuators.

VI. VALIDATION THROUGH ANSYS®

A rigid body model was built in ANSYS® with similar assumptions as in the analytical formulations with platforms treated as rigid bodies and struts as ideal springs. Simulation results for a $\langle 3, 3 \rangle$, $\langle 4, 4 \rangle$ and $\langle 3, 4 \rangle$ dynamically isotropic MGSP are listed in Table I. The closed-form solutions and simulation results are obtained for a typical payload with K =3748/6343, Q = 5.089*10^{-3} m^2, $m_p = 5$ Kg, and $k = 10^5$ N/m. The same geometric parameters are used for $\langle 3, 3 \rangle$ and $\langle 4, 4 \rangle$ $(n_1 = n_2)$ in simulations as obtained via closed-form. We obtained the dynamic isotropic index (DII), the ratio of the largest to the smallest natural frequency (DII should ideally be one or close to one), and the DII obtained from the simulation for all the MGSP closely matches our closed-form solution. This validates our closed-form analytical approach.

VII. CONCLUSION

This paper deals with the design of a class of dynamically isotropic MGSP. Previous researchers have mostly confined to two radii 6-6 MGSP. This work extends MGSP with arbitrary number of struts, and using a novel geometry-based approach, a complete closed-form solution in an explicit form is developed. The tangency condition for the virtual circle was found to yield a complete set of solutions. A new concept of a configuration transition point is discussed in detail. A 6-6 or $\langle 3, 3 \rangle$ MGSP is found most suitable for micro-vibration isolation among all other MGSPs due to its lowest natural frequency. The closed-form results for various MGSP or two-radii GSPs are successfully validated via simulation using an FE software. Our current work includes extending this study to account for centre of mass variation. The scope for future work consists of the incorporation of damping and active controls.

REFERENCES

[1] R. B. A. Shyam, N. Ahmad, R. Ranganath, and A. Ghosal, "Design of a dynamically isotropic Stewart-Gough platform for passive micro vibration isolation in spacecraft using optimization," *Journal of Spacecraft Technology*, vol. 30, no. 2, pp. 1-8, December 2019.

[2] Z. Tong, J. He, H. Jiang, and G Duan, "Optimal design of a class of generalized symmetric Gough–Stewart parallel manipulators with dynamic isotropy and singularity-free workspace," *Robotica*, vol. 30, no. 2, pp. 305–314, 2012.

[3] H. Yun, L. Liu, Q. Li, W. Li, and L. Tang, "Development of an isotropic Stewart platform for telescope secondary mirror," *Mechanical Systems and Signal Processing*, vol. 127, pp. 328–344, 2019.

[4] J. Hong-zhou, T. Zhi-zhong, and H. Jing-feng, "Dynamic isotropic design of a class of Gough–Stewart parallel manipulators lying on a circular hyperboloid of one sheet," *Mechanism and Machine Theory*, vol. 46, no. 3 , pp. 358–374, 2011.

[5] J. Yao, Y. Hou, H. Wang, T. Zhou, and Y. Zhao, "Spatially isotropic configuration of Stewart platform-based force sensor," *Mechanism and Machine Theory* , vol. 46, no. 3, pp. 142–155, 2011.

[6] Z. Tong, C. Gosselin, and H. Jiang, "Dynamic decoupling analysis and experiment based on a class of modified Gough-Stewart parallel manipulators with line orthogonality," *Mechanism and Machine Theory*, vol. 143, pp. 1-18 , 2020.

[7] H. Z. Jiang, J. F. He, Z. Z. Tong, and W. Wang, "Dynamic isotropic design for modified Gough–Stewart platforms lying on a pair of circular hyperboloids," *Mechanism and Machine Theory*, vol. 46, no. 9, pp. 1301–1315, 2011.

[8] Y. Yi, J.E. McInroy, and F. Jafari, "Generating classes of locally orthogonal Gough–Stewart platforms," *IEEE Transactions on Robotics*, vol. 21, no. 5, pp. 812-820, October 2005.

[9] A. Ghosal, "Robotics: Fundamental Concepts and Analysis," *Oxford University Press*, February 2006.

[10] A. A. Hanieh, "Active isolation and damping of vibrations via Stewart platform," *ULB Active Structures Laboratory*, pp. 62-66 , 2003.