SIAM J. NUMER. ANAL.
Vol. 51, No. 3, pp. 1392–1420

ANALYSIS OF A METHOD TO PARAMETERIZE PLANAR CURVES IMMERSED IN TRIANGULATIONS*

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Abstract. We prove that a planar C^2-regular boundary Γ can always be parameterized with its closest point projection π over a certain collection of edges Γ_h in an ambient triangulation by making simple assumptions on the background mesh. For Γ_h, we select the edges that have both vertices on one side of Γ and belong to a triangle that has a vertex on the other side. By imposing restrictions on the size of triangles near the curve and by requesting that certain angles in the mesh be strictly acute, we prove that $\pi : \Gamma_h \to \Gamma$ is a homeomorphism and that it is C^1 on each edge in Γ_h, and we provide bounds for the Jacobian of the parameterization. The assumptions on the background mesh are both easy to satisfy in practice and conveniently verified in computer implementations. The parameterization analyzed here was previously proposed by the authors and applied to the construction of high-order curved finite elements on a class of planar piecewise C^2-curves.

Key words. curve parameterization, closest point projection, curved finite elements

AMS subject classifications. 68U05, 65D18

DOI. 10.1137/110831805

1. Introduction. The purpose of this article is to analyze a method to parameterize planar C^2-regular boundaries over a collection of edges in a background triangulation. Such a parameterization was introduced by the authors in [14]. The method consists in making specific choices for the edges in the background mesh and for the map from these edges onto the curve. For the edges, we select the ones that have both vertices on one side of the (orientable) curve to be parameterized and belong to a triangle that has a vertex on the other side, as illustrated in Figure 1.1. Such edges are termed positive edges. For the map, we select the closest point projection of the curve. In this article, we prove that the closest point projection restricted to the collection of positive edges is a homeomorphism onto the curve and that it is C^1 on each positive edge (Theorem 3.1). For this, we have to impose restrictions on the size of a few triangles near the curve and request that certain angles in the background mesh be strictly smaller than 90°. We also compute bounds for the Jacobian of the resulting parameterization for the curve.

It is perhaps common knowledge that a sufficiently smooth curve can be parameterized with its closest point projection over the collection of interpolating edges in an adequately refined conforming triangulation. With Theorem 3.1, we generalize such an intuitive parameterization to also include nonconforming background meshes. In place of the interpolating edges in a conforming mesh, we pick the collection of positive edges in a nonconforming one, while still adopting the closest point projection to parameterize the curve. However, regularity for the curve and refinement for the mesh do not suffice. We also require certain angles in the mesh to be strictly acute, as depicted in Figure 1.1. In practice, such an assumption is easy to both

*Received by the editors April 22, 2011; accepted for publication (in revised form) February 4, 2013; published electronically May 2, 2013.
http://www.siam.org/journals/sinum/51-3/83180.html

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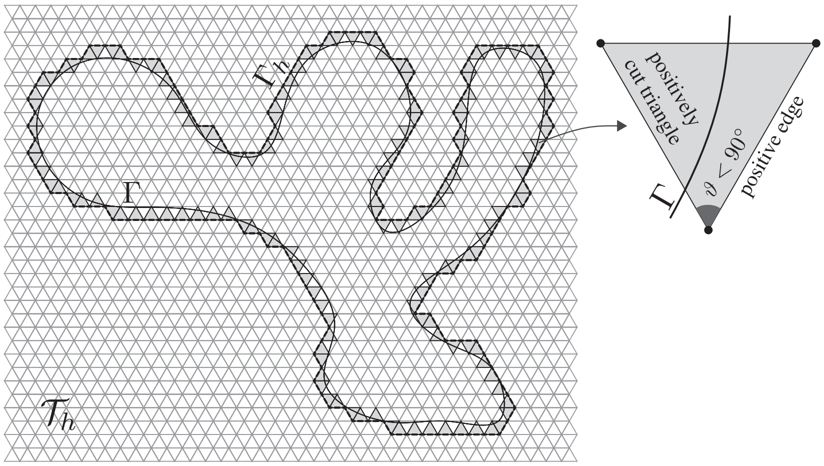

FIG. 1.1. *Illustration of the choice of edges in an ambient triangulation used to parameterize a C^2-regular boundary. The curve Γ is a cubic spline and is immersed in a nonconforming mesh of equilateral triangles. Triangles having one vertex inside the region enclosed by Γ and two vertices outside are said to be positively cut and are shaded in gray. The edge of such a triangle joining its two vertices outside is called a positive edge; their union is denoted by Γ_h and is drawn in dotted black lines. Theorem* 3.1 *identifies sufficient conditions for $\pi : \Gamma_h \to \Gamma$ to constitute a parameterization for Γ, where π is its closest point projection. Critical among these conditions is that a specific angle in each positively cut triangle be strictly acute, namely, the one at the vertex of the positive edge closest to Γ, as illustrated in the triangle on the right.*

check and satisfy. It is perhaps surprising that a local algebraic condition on angles in triangles near the curve precipitates a global topological result, more so because the angles required to be acute are irrelevant in the parameterization itself— neither the identification of positive edges nor the mapping onto the curve (the closest point projection) depends on them.

A compelling consequence of Theorem 3.1 is that *any* planar smooth boundary can be parameterized with its closest point projection over the collection of positive edges in *any* sufficiently refined background mesh of equilateral triangles. It is also interesting to note that the theorem does not guarantee the same with a background mesh of right-angled triangles. Such meshes may not satisfy the required assumption on angles; see (3.1b) in Theorem 3.1. On a related note, in [13, 14] we describe a way of parameterizing curves over edges and diagonals of meshes of parallelograms, which in particular includes structured meshes of rectangles. See also [3] for a triangulation algorithm with a similar objective.

The parameterization studied is independent of the particular description adopted for the curve, is easy to implement, and is readily parallelizable. It also extends naturally to planar curves with endpoints, corners, self-intersections, T-junctions, and practically all planar curves of interest in engineering and computer graphics applications; see [14] and [13, Chapter 4]. The idea is to construct such curves by splicing arcs of C^2-regular boundaries and parameterize each arc with its closest point projection.

One of the main motivations behind the parameterization over positive edges is to accurately represent planar curved domains over nonconforming background meshes. Once the curved boundary is parameterized over a collection of nearby edges, we show in [13, Chapter 5] how a suitable collection of triangles in the background mesh can be mapped to curved ones to yield an exact spatial discretization for the curved domain. The construction of such mappings from straight triangles to curved ones and their analysis in the context of high-order finite elements with optimal convergence properties has been the subject of numerous articles; we refer to a representative few [4, 5, 7, 11, 12, 15, 16] for details on this subject. Almost without exception, these constructions have two assumptions in common: (i) a mesh with edges that interpolate the curved boundary and (ii) a (local) parametric representation for the curved boundary. The former entails careful mesh generation, while the latter is a strong assumption on how the boundary is described. The parameterization analyzed here enables relaxing both these assumptions.

An outline of the proof of Theorem 3.1 is given in section 3.3. The crux of the proof is demonstrating injectivity of the closest point projection (π) over the collection of positive edges (Γ_h). Regularity of the parameterization and estimates for the Jacobian follow easily from regularity of the curve (Γ) and some straightforward calculations. We prove injectivity by inspecting the restriction of π to each positive edge, then to pairs of intersecting positive edges, and finally to connected components of Γ_h. Requesting specific angles in the mesh to be acute has a simple geometric motivation (see Figure 3.1) and ensures injectivity over each positive edge (section A, section 4). Extending this to the entire set Γ_h is nontrivial, requiring some careful, albeit simple, topological arguments. It entails understanding how and how many positive edges intersect at each vertex in Γ_h, leading us to show in section 5 that each connected component of Γ_h is a Jordan curve. We then show in section 6 that the restriction of π to each connected component of Γ_h is a parameterization of a connected component of Γ. Finally, in section 7 we establish a correspondence between connected components of Γ and Γ_h.

2. Preliminary definitions. In order to state our main result with the requisite assumptions, a few definitions are essential. First, we define the family of planar C^2-regular boundaries, the curves we consider for parameterization.

Definition 2.1 (see [8, Definition 1.2]). *A bounded open set $\Omega \subset \mathbb{R}^2$ has a C^2-regular boundary if there exists $\Psi \in C^2\left(\mathbb{R}^2, \mathbb{R}\right)$ such that $\Omega = \left\{x \in \mathbb{R}^2 : \Psi(x) < 0\right\}$ and $\Psi(x) = 0$ implies $|\nabla\Psi| \geq 1$. We say that Ω is a C^2-regular domain and that $\partial\Omega$ is a C^2-regular boundary. The function Ψ is called a defining function for Ω.*

There are a few equivalent notions of C^2-regular boundaries (and more generally C^k-regular boundaries); see [9]. For future reference, we note that each connected component of a C^2-regular boundary is a Jordan curve with bounded curvature.

We recall the definitions of the signed distance function and the closest point projection for a curve Γ that is the boundary of an open and bounded set Ω in $\mathbb{R}^2$. The signed distance to Γ is the map $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as $-\min_{y\in\Gamma} d(\cdot, y)$ over Ω and as $\min_{y\in\Gamma} d(\cdot, y)$ elsewhere. The function $d(\cdot, \cdot)$ is the Euclidean distance in $\mathbb{R}^2$. The closest point projection π onto Γ is the map $\pi : \mathbb{R}^2 \to \Gamma$ given by $\pi(\cdot) = \arg\min_{y\in\Gamma} d(\cdot, y)$.

The following theorem quoted from [8] is a vital result for our analysis. It concerns the regularity of the maps ϕ and π for a C^2-regular boundary. The theorem also shows that ϕ is a defining function for a C^2-regular domain. In the statement, the ε-ball centered at $x \in \mathbb{R}^2$ is the set $B(x, \varepsilon) := \{y : d(x, y) < \varepsilon\}$ and the ε-neighborhood of $A \subset \mathbb{R}^2$ is the set $B(A, \varepsilon) := \cup_{x\in A} B(x, \varepsilon)$.

Theorem 2.2 (see [8, Theorem 1.5]). *If $\Omega \subset \mathbb{R}^2$ is an open set with a C^2-regular boundary, then there exists $r_n > 0$ such that $\phi\colon B(\partial\Omega, r_n) \to (-r_n, r_n)$ and $\pi\colon B(\partial\Omega, r_n) \to \partial\Omega$ are well defined. The map ϕ is C^2, while π is a C^1 retraction onto $\partial\Omega$. The mapping $x \mapsto (\phi(x), \pi(x)) : B(\partial\Omega, r_n) \to (-r_n, r_n) \times \partial\Omega$ is a C^1-diffeomorphism* *with inverse* $(\phi, \xi) \mapsto \xi + \phi \hat{N}(\xi) : (-r_n, r_n) \times \partial\Omega \to B(\partial\Omega, r_n)$*, where* $\hat{N}(\xi)$ *is the unit outward normal to* $\partial\Omega$ *at* ξ*. Furthermore,* ϕ *is the unique solution of* $|\nabla\phi| = 1$ *in* $B(\partial\Omega, r_n)$ *with* $\phi = 0$ *on* $\partial\Omega$ *and* $\nabla\phi \cdot \hat{N} > 0$ *on* $\partial\Omega$*.*

In Theorem 2.2, by saying that ϕ and π are well defined over $B(\partial\Omega, r_n)$, we mean that these maps are defined and have a unique value at each point in $B(\partial\Omega, r_n)$. The following proposition follows from [6, section 14.6]. A simple derivation specific to planar curves can be found in [14].

PROPOSITION 2.3. *Let* $\Gamma \subset \mathbb{R}^2$ *be a* C^2*-regular boundary with signed distance function* ϕ*, closest point projection* π*, signed curvature* κ_s*, and unit tangent* $\hat{T}$*. If* $p \in B(\Gamma, r_n)$ *and* $|\phi(p)\kappa_s(\pi(p))| < 1$*, then*

$$\nabla\pi(p) = \frac{\hat{T}(\pi(p)) \otimes \hat{T}(\pi(p))}{1 - \phi(p)\,\kappa_s(\pi(p))} \tag{2.1a}$$

$$\text{and} \quad \nabla\nabla\phi(p) = -\kappa_s(\pi(p))\nabla\pi(p). \tag{2.1b}$$

For parameterizing C^2-regular boundaries, we will consider background meshes that are triangulations of polygonal domains (cf. [10, Chapter 4]). We mention the related terminology and notation used in the remainder of the article. With triangulation $\mathcal{T}_h$, we associate a pairing (V, C) of a vertex list V that is a finite set of points in $\mathbb{R}^2$ and a connectivity table C that is a collection of ordered 3-tuples in $V \times V \times V$ modulo permutations. A vertex in $\mathcal{T}_h$ is thus an element of V (and hence a point in $\mathbb{R}^2$). An edge in $\mathcal{T}_h$ is a closed line segment joining two vertices of a member of C. The relative interior of an edge e_{pq} with endpoints (or vertices) p and q is the set $\mathbf{ri}\,(e_{pq}) = e_{pq} \setminus \{p, q\}$.

A triangle K in $\mathcal{T}_h$, denoted $K \in \mathcal{T}_h$, is the interior of the triangle in $\mathbb{R}^2$ with vertices given by its connectivity $\hat{K} \in C$. Frequently, we will not distinguish between K and $\hat{K}$ unless the distinction is essential. We refer to the diameter of K by h_K and the diameter of the largest ball contained in $\overline{K}$ by ρ_K. The ratio $\sigma_K := h_K/\rho_K$ is called the shape parameter of K [10, Chapter 3]. Later, we will invoke the fact that $\sigma_K \geq \sqrt{3}$ with equality holding for equilateral triangles.

To consider curves immersed in background triangulations, we introduce the following terminology.

DEFINITION 2.4. *Let* $\Gamma \subset \mathbb{R}^2$ *be a* C^2*-regular boundary with signed distance function* ϕ *and let* $\mathcal{T}_h$ *be a triangulation of a polygon in* $\mathbb{R}^2$*.*

(i) *We say that* Γ *is immersed in* $\mathcal{T}_h$ *if* $\Gamma \subset \operatorname{int}\left(\cup_{K \in \mathcal{T}_h} \overline{K}\right)$*.*

(ii) *A triangle in* $\mathcal{T}_h$ *is positively cut by* Γ *if* $\phi \geq 0$ *at precisely two of its vertices.*

(iii) *An edge in* $\mathcal{T}_h$ *is a positive edge if* $\phi \geq 0$ *at both of its vertices and if it is an edge of a triangle that is positively cut by* Γ*.*

(iv) *The proximal vertex of a triangle positively cut by* Γ *is the vertex of its positive edge closest to* Γ*. When both vertices of the positive edge are equidistant from* Γ*, the one containing the smaller interior angle is designated to be the proximal vertex. If the angles are equal as well, either vertex of the positive edge can be assigned as the proximal vertex.*

(v) *The conditioning angle of a triangle positively cut by* Γ *is the interior angle at its proximal vertex.*

(vi) *Let* $K, K^{adj} \in \mathcal{T}_h$ *be such that* K *is positively cut by* Γ*,* K *has positive edge* e*,* $e \cap \Gamma \neq \emptyset$*, and* $\overline{K} \cap \overline{K^{adj}} = e$*. Then, the angle adjacent to the positive edge of* K*, denoted* ϑ_K^{adj}*, is defined as the minimum of the interior angles in* K^{adj} *at the vertices of* e*.*

3. Main result.

The main result of this article is the following.

THEOREM 3.1. *Consider a C^2-regular boundary $\Gamma \subset \mathbb{R}^2$ with signed distance function ϕ, closest point projection π, and curvature κ. Let Γ be immersed in a triangulation $\mathcal{T}_h$. Denote the union of positive edges in $\mathcal{T}_h$ by Γ_h and the collection of triangles positively cut by Γ in $\mathcal{T}_h$ by $\mathcal{P}_h$. For each $K \in \mathcal{P}_h$, let*

$$\vartheta_K := \text{conditioning angle of } K,$$

$$\vartheta_K^{adj} := \text{angle adjacent to positive edge of } K \text{ when defined},$$

$$M_K := \max_{B(K,h_K)\cap\Gamma} \kappa \quad \text{and} \quad C_K^h := \frac{M_K}{1 - M_K h_K}.$$

Assume that for each connected component γ of Γ, $\gamma_h := \{x \in \Gamma_h \,:\, \pi(x) \in \gamma\} \neq \emptyset$. If for each $K \in \mathcal{P}_h$ we have

$$h_K < r_n, \tag{3.1a}$$

$$\vartheta_K < 90^\circ, \tag{3.1b}$$

$$0 < \sigma_K C_K^h h_K < \min\left\{\cos\vartheta_K, \sin\frac{\vartheta_K}{2}\right\}, \tag{3.1c}$$

$$\text{and} \quad C_K^h h_K < \frac{1}{2}\sin\vartheta_K^{adj} \quad \text{whenever } \vartheta_K^{adj} \text{ is defined}, \tag{3.1d}$$

then

(i) *each positive edge in Γ_h is an edge of precisely one triangle in $\mathcal{P}_h$,*
(ii) *for each positive edge $e \subset \Gamma_h$, π is a C^1-diffeomorphism over $\boldsymbol{ri}(e)$,*
(iii) *if $K = (p,q,r) \in \mathcal{P}_h$ has positive edge e_{pq}, then*

$$-C_K^h h_K^2 < \phi(x) \leq h_K \quad \forall x \in e_{pq}. \tag{3.2}$$

The Jacobian J of the map $\pi : \boldsymbol{ri}(e_{pq}) \to \Gamma$ satisfies

$$0 < \frac{\sin(\beta_K - \vartheta_K)}{1 + M_K h_K} \leq J(x) = \left|\nabla\pi(x)\cdot\frac{(p-q)}{d(p,q)}\right| \leq \frac{1}{1 - M_K h_K} \quad \forall x \in \boldsymbol{ri}(e_{pq}), \tag{3.3}$$

$$\text{where} \quad \cos\beta_K := C_K^h\sigma_K h_K - \eta_K,\ \ \beta_K \in [0^\circ, 180^\circ], \tag{3.4}$$

$$\eta_K := \frac{\min\{\phi(p),\phi(q)\} - \phi(r)}{h_K}. \tag{3.5}$$

(iv) *The map $\pi : \Gamma_h \to \Gamma$ is a homeomorphism. In particular, γ_h as defined above is a simple, closed curve.*

3.1. Discussion of the statement.

With Γ and Γ_h as in the statement, Theorem 3.1 asserts sufficient conditions under which $\pi : \Gamma_h \to \Gamma$ is a homeomorphism. The statement of the theorem extends also to the case when edges in Γ_h are identified using the function $-\phi$ instead of ϕ. This corresponds to selecting the collection of *negative* edges for parameterizing Γ. Of course, a different collection of angles are required to be acute. If triangles in the vicinity of the curve are all acute angled, the theorem shows that there are two disjoint collections of edges homeomorphic to Γ.

We make two important assumptions on the background mesh; we briefly examine them and discuss how they can be satisfied in practice in section 3.2. The first assumption is, expectedly, on the size of triangles near Γ, as conveyed by conditions

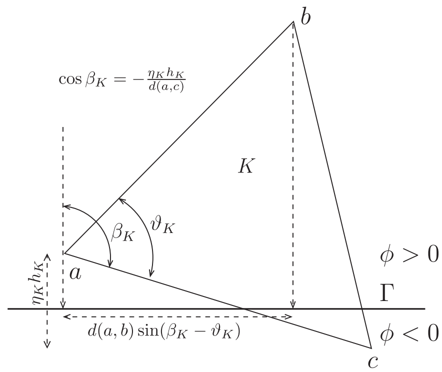

Fig. 3.1. *Illustration to explain the rationale behind the acute conditioning angle assumption. Triangle K is positively cut by Γ. Although $\beta_K > 90°$, it can be arbitrarily close to $90°$ by changing the locations of vertices b and c. Requesting $\vartheta_K < 90°$ ensures that $\beta_K - \vartheta_K > 0°$ always and hence that $\pi(e_{ab})$ has nonzero length.*

(3.1a), (3.1c), and (3.1d). For instance, if the mesh size is too large, then π may not even be single valued over Γ_h.

Assumption (3.1b), which we term the *acute conditioning angle assumption*, is perhaps less intuitive. Once the set Γ_h has been identified, the angles that positive edges make with other edges in the background mesh $\mathcal{T}_h$ are irrelevant. Rather, the rationale behind (3.1b) is that it provides a means to control the orientation of positive edges with respect to local normals to the curve. We explain this idea in section 3.1.1 using a simple example.

It is worth emphasizing that the assumptions on the background mesh in (3.1) are not very restrictive principally because there is no conformity required with Γ. Besides, the region triangulated by $\mathcal{T}_h$ can be quite arbitrary and need only contain Γ in the sense of Definition 2.4(i). In particular, while considering ambient triangulations of larger sets, the restrictions on the size, quality, and angles stemming from (3.1) apply only to a subset of the collection of triangles intersected by Γ, namely, positively cut triangles and triangles having positive edges that are intersected by Γ.

Finally, we mention that Theorem 3.1 guarantees a parameterization for Γ provided the collection of triangles positively cut by each of its connected components is nonempty. This is apparent from the fact that all restrictions on the mesh size and angles in (3.1) apply only to positively cut triangles and triangles having positive edges that are intersected by Γ. For instance, if a connected component γ of Γ is contained in the interior of a triangle in $\mathcal{T}_h$, then no triangle is positively cut by it. Of course, it is possible for the collection of triangles positively cut by γ to be empty in a multitude of ways. In principle, sufficient conditions are easily identified to ensure that at least one triangle is positively cut by each connected component of Γ. In practice, however, it is much simpler to inspect the sign of ϕ at the vertices of triangles and verify the presence of positively cut triangles rather than check such conditions.

3.1.1. The acute conditioning angle assumption. Consider a locally straight curve Γ as shown in Figure 3.1. Triangle K shown in the figure is positively cut by Γ and has positive edge e_{ab} and proximal vertex a. Abusing the definition in (3.4), we have $\cos\beta_K = -\eta_K h_K/d(a,c)$ as indicated in the figure. (The two definitions coincide if the length of the edge e_{ac} is h_K.) The projection of e_{ab} onto Γ has length $d(a,b)\sin(\beta_K - \vartheta_K)$. For π to be injective over e_{ab}, we need to ensure that

$0^\circ < \beta_K - \vartheta_K < 180^\circ$. Even though the angle β_K depicted in the figure is strictly larger than 90°, it can be made arbitrarily close to 90° by altering the locations of vertices a and c. Therefore, we request that the conditioning angle ϑ_K be smaller than 90°, thereby ensuring $\beta_K - \vartheta_K > 0^\circ$. The assumptions $\phi(a) \leq \phi(b)$ and $\vartheta_K < 90^\circ$ together imply that $\beta_K - \vartheta_K < 180^\circ$.

We refer to [13, 14] for simple examples in which π fails to be injective over Γ_h because conditioning angles fails to be acute. Of course, (3.1b) is only a sufficient condition for injectivity. In fact, a simple way to relax assumption (3.1b) is by defining an equivalence relation $\overset{\Gamma}{\simeq}$ over the family of triangulations in which Γ is immersed. Consider two triangulations $\mathcal{T}_h = (V, C)$ and $\mathcal{T}'_{h'} = (V', C')$. We say $\mathcal{T}_h \overset{\Gamma}{\simeq} \mathcal{T}'_{h'}$ if there is a bijection $\Phi : V \to V'$ such that

(i) $(p, q, r) \in C \iff (\Phi(p), \Phi(q), \Phi(r)) \in C'$,
(ii) $\phi(v) \geq 0 \iff \phi(\Phi(v)) \geq 0$,
(iii) $\phi(v) < 0 \iff \phi(\Phi(v)) < 0$,
(iv) $v \in \Gamma_h \Rightarrow \Phi(v) = v$.

The map Φ can be interpreted as a (constrained) perturbation of vertices in $\mathcal{T}_h$ to yield a new mesh $\mathcal{T}'_{h'}$. It is clear from the definition of the equivalence relation that both $\mathcal{T}_h$ and $\mathcal{T}'_{h'}$ have *exactly the same set of positive edges* even though their positively cut triangles can have very different conditioning angles. The key point is that the result of the theorem can be applied to $\mathcal{T}_h$ from merely knowing the existence of a triangulation in its equivalence class that has acute conditioning angles. In light of this observation, the theorem applies even to some families of background meshes that do not satisfy assumption (3.1b).

With no conformity requirements on the background mesh, the acute conditioning angle assumption (3.1b) is easy to satisfy in practice. A simple way, for example, is to ensure that triangles in the vicinity of Γ_h in the background mesh are acute angled; even simpler is to use background meshes consisting of all acute angled triangles. Such acute triangulations, including adaptively refined ones, are conveniently constructed by tiling quadtrees using stencils of acute angled triangles provided in [2].

3.2. Restrictions on triangle sizes. Conditions restricting the mesh size, namely, (3.1a), (3.1c), and (3.1d), were identified by simply tracking the restrictions on the mesh size in the proof of Theorem 3.1. They are easily checked for a given curve and background mesh and can be used to guide refinement of background meshes near the boundaries of domains. Furthermore, they make transparent which parameters related to the curve and to the mesh influence how much refinement is required. For instance, (3.1a) shows that a finer mesh is required if the curve has small features. The requirement that $\sigma_K C_K^h h_K$ be positive in (3.1c) is equivalent to $M_K h_K < 1$, which reveals that smaller triangles are required where the curve has large curvature. More refinement is also needed when conditioning angles are close to 90°, and when triangles are poorly shaped (indicated by large values of σ_K or small values of ϑ_K^{adj}).

Commonly used meshing algorithms usually guarantee *shape regularity* and bounds for interior angles in triangles with mesh refinement. Consequently, there exist mesh size independent constants $\sigma > 0$ and $0^\circ < \theta_{\min} \leq \theta_{\max} < 180^\circ$ such that the shape parameter is bounded by σ and interior angles of triangles are bounded between $\theta_{\min}$ and $\theta_{\max}$. As discussed above, conditioning angles can be guaranteed to be acute independent of the mesh size, for example, $\vartheta_K = 60^\circ$ for background meshes of equilateral triangles. Angles in triangulations constructed using stencils in [2] are guaranteed to lie between 36° and 80°.

It is imperative also to consider if the requirements on the mesh size posed by (3.1) are too conservative. We check this for a specific example of a circle of radius R immersed in a background mesh of equilateral triangles. In such a case, we have $h_K = h$ for each triangle in the mesh and $\vartheta_K = 60°, \sigma_K = \sqrt{3}, r_n = M_K = 1/R$, and $\vartheta_K^{\text{adj}} = 60°$ (when defined) for each positively cut triangle K. Then, satisfying (3.1) requires $h < h_0 := R/(1 + 2\sqrt{3}) \simeq 0.224R$. The a priori estimate $h_0 = 0.224R$ is a reasonable one because it is comparable to R. Of course, the estimate will change with the choice of background meshes.

3.2.1. Bound for the Jacobian. Equation (3.3) provides an estimate for the Jacobian of the parameterization. Inspecting the lower bound in (3.3), which is the critical one, shows that $J \geq \sin(\beta_K - \vartheta_K)$ if $M_K h_K = 0$. This is precisely the Jacobian computed for a line, as in Figure 3.1, when the definition of β_K in (3.4) is replaced by that in the figure. The same interpretation of the lower bound holds when $M_K \neq 0$ but h_K is small. In this case, each positive edge parameterizes a small subset of Γ, which appears essentially straight.

For reasonably large values of $M_K h_K$, the angle β_K in (3.4) can be close to 90°, even acute. Hence $\beta_K - \vartheta_K$ can be small. In light of this, we mention that a smaller conditioning angle yields a better parameterization, one with J closer to 1. Finally, with mesh size independent bounds for σ_K and ϑ_K, it is straightforward to demonstrate that the estimates for J in (3.3) are in turn bounded away from zero independent of the mesh size. (Specifically, $\beta_K - \vartheta_K$ and $1 \pm M_K h_K$ appearing in the estimate can be bounded independent of the mesh size.)

3.3. Outline of proof. We briefly discuss the outline of the proof of Theorem 3.1. The critical step is showing that π is injective over Γ_h. To this end, we proceed in simple steps by considering the restriction of π over each positive edge, then over pairs of intersecting positive edges, and finally over connected components of Γ_h.

In Appendix A, we compute bounds for the signed distance function ϕ on Γ_h and for angles between positive edges and local tangents/normals to Γ. By requiring that the size of positively cut triangles be sufficiently small and by invoking assumption (3.1b), we show that a positive edge is never parallel to a local normal to Γ (Proposition 4.1). From here, we infer that π is injective over each positive edge (Lemma 4.2). The required bounds for the Jacobian in (3.3) also follow easily from the angle estimates. Part (ii) of the theorem is then a direct consequence of the inverse function theorem.

A logical next step is to show that π is injective over each pair of intersecting positive edges (Proposition 6.2). For this, in section 5 we first examine how positive edges intersect. Lemma 5.2 states that precisely two positive edges intersect at each vertex in Γ_h. This result leads us to conclude that Γ_h is in fact a collection of simple, closed curves (Lemma 5.3).

Knowing that (i) π is injective over each pair of intersecting positive edges, (ii) each connected component of Γ_h is a simple, closed curve, and (iii) π is continuous over Γ_h, we demonstrate (in Lemma 6.1) that π is a homeomorphism over each connected component of Γ_h. What remains to be shown is that precisely one connected component of Γ_h is mapped to each connected component of Γ. We do this in section 7 by illustrating that the collection of positive edges that map to a connected component of Γ is itself a connected set (Lemma 7.1).

3.4. Assumptions and notation for subsequent sections. In all results stated in subsequent sections, we presume that all of (3.1) in the statement of

Theorem 3.1 holds. In several intermediate results, one or more of these assumptions could be relaxed.

We shall denote the unit normal and unit tangent to Γ at $\xi \in \Gamma$ by $\hat{N}(\xi)$ and $\hat{T}(\xi)$, respectively. We assume an orientation for Γ such that $\hat{N} = \nabla\phi$ on the curve and that $\{\hat{T}, \hat{N}\}$ constitutes a right-handed basis for $\mathbb{R}^2$ at any point on the curve. Given distinct points $a, b \in \mathbb{R}^2$, we denote the unit vector pointing from a to b by $\hat{U}_{ab}$ and define $\hat{U}_{ab}^{\perp}$ such that $\{\hat{U}_{ab}, \hat{U}_{ab}^{\perp}\}$ is a right-handed basis.

The following simple calculation establishes the ranges of parameters η_K and β_K introduced in the statement of Theorem 3.1. Furthermore, for each $K \in \mathcal{P}_h$, part (ii) of Proposition 3.2 together with (3.1a) implies that $\overline{K} \subset B(\Gamma, r_n)$. Then Theorem 2.2 shows that π is C^1 and in particular is well defined over $\overline{K}$. Since any positive edge is an edge of some triangle in $\mathcal{P}_h$, we get that $\Gamma_h \subset B(\Gamma, r_n)$ and hence that π is well defined and continuous on Γ_h. We shall frequently use these consequences of the proposition in the remainder of the article, often without explicitly referring to them.

PROPOSITION 3.2. *Let $K = (a, b, c) \in \mathcal{P}_h$. Then*
- (i) $\overline{K} \cap \Gamma \neq \emptyset$, *in particular,* $\phi(c) < 0 \Rightarrow e_{bc} \cap \Gamma \neq \emptyset, e_{ac} \cap \Gamma \neq \emptyset$;
- (ii) $|\phi| \leq h_K$ *on* $\overline{K}$;
- (iii) η_K *defined in* (3.5) *satisfies* $0 < \eta_K \leq 1$;
- (iv) β_K *given by* (3.4) *is well defined and* $\beta_K > \vartheta_K$.

Proof. We only show (iv) and the upper bound in (iii), since the others follow directly from the definitions. To this end, assume that $\phi(c) < 0$ and $\phi(a), \phi(b) \geq 0$, and consider any $\xi \in e_{ac} \cap \Gamma$. From the definition of η_K in (3.5), we have

$$\eta_K h_K \leq \phi(a) - \phi(c) \leq d(a, \Gamma) + d(c, \Gamma) \leq d(a, \xi) + d(c, \xi) \leq h_K,$$

which shows that $\eta_K \leq 1$. To show that β_K is well defined, we check that $\cos\beta_K \in [-1, 1]$. Noting that $\sigma_K C_K^h h_K \geq 0$ and $\eta_K \leq 1$ shows that

$$\cos\beta_K = \sigma_K C_K^h h_K - \eta_K \geq -\eta_K \geq -1.$$

For the upper bound, we have

$$\begin{aligned}\cos\beta_K &= \sigma_K C_K h_K - \eta_K,\\ &\leq \sigma_K C_K h_K \qquad (\text{using } \eta_K > 0)\,,\\ &\leq \cos\vartheta_K \leq 1 \qquad (\text{from (3.1c)})\,,\end{aligned}$$

which also shows that $\beta_K > \vartheta_K$. ∎

4. Injectivity on each positive edge. To show the injectivity of π on each positive edge (Lemma 4.2) and estimate the Jacobian of this mapping (Lemma 4.5), we essentially follow the calculation illustrated in Figure 3.1. In both arguments, we use the following angle estimate that is proved in Appendix A.

PROPOSITION 4.1. *Let $K = (a, b, c) \in \mathcal{P}_h$ have positive edge e_{ab} and proximal vertex a. Then*

$$-\frac{3}{2} C_K^h h_K \leq \hat{N}(\pi(x)) \cdot \hat{U}_{ab} \leq \cos(\beta_K - \vartheta_K) \quad \forall x \in e_{ab}. \tag{4.1}$$

In particular, $|\hat{N}(\pi(x)) \cdot \hat{U}_{ab}| < 1$ *and* $|\hat{T}(\pi(x)) \cdot \hat{U}_{ab}| > 0$.

LEMMA 4.2. *The restriction of π to each positive edge in Γ_h is injective.*

Proof. Let $(a, b, c) \in \mathcal{P}_h$ have positive edge e_{ab} and proximal vertex a. We proceed by contradiction. Suppose that $x, y \in e_{ab}$ are distinct points such that $\pi(x) = \pi(y)$. From Theorem 2.2 and $\pi(x) = \pi(y)$, we have

$$x = \pi(x) + \phi(x)\hat{N}(\pi(x)), \tag{4.2a}$$

$$y = \pi(y) + \phi(y)\hat{N}(\pi(y)) = \pi(x) + \phi(y)\hat{N}(\pi(x)). \tag{4.2b}$$

Noting $x \neq y$ in (4.2) implies that $\phi(x) \neq \phi(y)$. Therefore, subtracting (4.2b) from (4.2a) yields

$$\hat{N}(\pi(x)) = \frac{x - y}{\phi(x) - \phi(y)}. \tag{4.3}$$

By definition of $x, y \in e_{ab}$, $x - y$ is a vector parallel to $\hat{U}_{ab}$. Therefore (4.3) in fact shows that $|\hat{N}(\pi(x)) \cdot \hat{U}_{ab}| = 1$, contradicting Proposition 4.1. ▯

Before showing the bounds in (3.3) for the Jacobian, we prove Corollary 4.4, a useful step in showing part (iv) of Theorem 3.1. As discussed in section 3.4, continuity of π on each positive edge follows from part (ii) of Proposition 3.2. The continuity of its inverse is a consequence of Lemma 4.2 and the following result in basic topology, which we use here and later in section 6.

THEOREM 4.3 (see [1, Chapter 3]). *A one-to-one, onto, and continuous function from a compact space to a Hausdorff space is a homeomorphism.*

COROLLARY 4.4 (of Lemma 4.2). *Let e be a positive edge in Γ_h. Then $\pi : e \to \pi(e)$ is a homeomorphism.*

Proof. From part (ii) of Proposition 3.2 and Lemma 4.2, we know that π is continuous and injective on e. The corollary then follows from Theorem 4.3. ▯

LEMMA 4.5. *Let $K = (a, b, c) \in \mathcal{P}_h$ have positive edge e_{ab}. Then π is C^1 over $\mathbf{ri}(e_{ab})$ and*

$$0 < \frac{\sin(\beta_K - \vartheta_K)}{1 + M_K h_K} \leq \left|\nabla\pi(x) \cdot \hat{U}_{ab}\right| \leq \frac{1}{1 - M_K h_K} \leq \frac{5}{3} \quad \forall x \in e_{ab}. \tag{4.4}$$

Proof. From part (ii) of Proposition 3.2 and (3.1a), we know $e_{ab} \subset B(\Gamma, r_n)$. Then Theorem 2.2 shows that π is C^1 over $\mathbf{ri}(e_{ab})$.

Consider any $x \in \mathbf{ri}(e_{ab})$. Since $|\phi(x)| \leq h_K$ (Proposition 3.2),

$$\kappa(\pi(x)) \leq \max_{B(x,h_K)\cap\Gamma} \kappa \leq \max_{B(K,h_K)\cap\Gamma} \kappa = M_K. \tag{4.5}$$

Therefore, $|\phi(x)\kappa(\pi(x))| \leq M_K h_K$, which is smaller than 1 because of the assumption $\sigma_K C_K^h h_K > 0$ in (3.1c). Then from Proposition 2.3, we get

$$J(x) := \left|\nabla\pi(x) \cdot \hat{U}_{ab}\right| = \frac{|\hat{U}_{ab} \cdot \hat{T}(\pi(x))|}{|1 - \phi(x)\kappa_s(\pi(x))|}, \tag{4.6}$$

where κ_s is the signed curvature of Γ (and $\kappa = |\kappa_s|$). From $|\phi(x)\kappa_s(\pi(x))| \leq M_K h_K < 1$, we get

$$1 - M_K h_K \leq |1 - \phi(x)\kappa_s(\pi(x))| \leq 1 + M_K h_K. \tag{4.7}$$

From Proposition 4.1, we have

$$|\sin(\beta_K - \vartheta_K)| \leq \left|\hat{T}(\pi(x)) \cdot \hat{U}_{ab}\right| \leq 1. \tag{4.8}$$

Note, however, from part (iv) of Proposition 3.2 that $\beta_K > \vartheta_K \Rightarrow |\sin(\beta_K - \vartheta_K)| = \sin(\beta_K - \vartheta_K)$. Then using (4.7) and (4.8) in (4.6) yields the lower and upper bounds for $|J(x)|$ in (4.4).

It remains to show that these bounds are meaningful, i.e., the lower bound is positive and the upper bound is not arbitrarily large. The former is a consequence of $\beta_K > \vartheta_K$ (from Proposition 3.2). We know from (3.1c) that $\sigma_K C_K^h h_K < \sin(\vartheta_K/2) < 1$. Then, using $M_K h_K < 1$ from (3.1c) and $\sigma_K \geq \sqrt{3}$, we get $M_K h_K < (1+\sqrt{3})^{-1} < 2/5$, which renders the upper bound in (4.4) independent of h_K. ∎

By using the strictly positive lower bound for the Jacobian in the inverse function theorem, we conclude that $\pi : \mathbf{ri}(e) \to \Gamma$ is a local C^1-diffeomorphism on each positive edge e. Since we have already shown the injectivity of this map in Lemma 4.2, part (ii) of Theorem 3.1 follows.

5. The set $\boldsymbol{\Gamma_h}$. An essential step in showing that π is injective over Γ_h is understanding how positive edges intersect. The goal of this section is to demonstrate that Γ_h is a union of simple, closed curves (Lemma 5.3). We achieve this by considering how many positive edges intersect at each vertex in Γ_h. In Lemma 5.2, we state that this number is precisely two. Additionally, as claimed in part (i) of Theorem 3.1 and stated below in Lemma 5.1, each positive edge belongs to precisely one positively cut triangle. The proofs of these two lemmas are somewhat laborious and hence are included in Appendix B.

LEMMA 5.1. *Each positive edge in $\mathcal{T}_h$ is a positive edge of precisely one triangle positively cut by Γ.*

LEMMA 5.2. *Precisely two distinct positive edges intersect at each vertex in Γ_h.*

LEMMA 5.3. *Let γ_h be a connected component of Γ_h. Then γ_h is a simple, closed curve that can be represented as*

$$\gamma_h = \bigcup_{i=0}^{n} e_{v_i v_{(i+1) mod\,(n)}}, \tag{5.1}$$

where $v_0, \ldots, v_n$ are all the distinct vertices in γ_h and $2 \leq n < \infty$.

Proof. We will only prove (5.1). That γ_h is a simple and closed curve follows immediately from such a representation.

Denote the number of vertices in γ_h by $n+1$ for some integer n. Since γ_h is nonempty, it contains at least one positive edge, say, $e_{v_0 v_1}$ with vertices v_0 and v_1. Lemma 5.2 shows that precisely two positive edges intersect at v_1. Therefore, we can find vertex $v_2 \in \gamma_h$ different from v_0, v_1 such that $e_{v_1 v_2}$ a positive edge. This shows that $n \geq 2$. Of course $n < \infty$ because there are only finitely many vertices in $\mathcal{T}_h$.

We have identified vertices v_0, v_1, and v_2 such that $e_{v_0 v_1}, e_{v_1 v_2} \subset \gamma_h$. Suppose that we have identified vertices $v_0, v_1, \ldots, v_{k-1}$ for $k \in \{2, \ldots, n\}$ such that $e_{v_i v_{(i+1)}} \subset \gamma_h$ for each $0 \leq i \leq k-2$. We show how to identify vertex v_k such that $e_{v_{(k-1)} v_k} \subset \gamma_h$. Lemma 5.2 shows that precisely two positive edges intersect at $v_{(k-1)}$. One of them is $e_{v_{(k-2)} v_{(k-1)}}$. Let v_k be such that $e_{v_{(k-1)} v_k}$ is the other positive edge. While v_k is different from $v_{(k-2)}$ and $v_{(k-1)}$ by definition, it remains to be shown that $v_k \neq v_i$ for $0 \leq i < k-2$. To this end, note that for $1 \leq i < k-2$, we have already found two positive edges that intersect at v_i, namely, $e_{v_{(i-1)} v_i}$ and $e_{v_i v_{(i+1)}}$. Therefore, it follows from Lemma 5.2 that $e_{v_i v_{(k-1)}}$ cannot be a positive edge for $1 \leq i < k-2$. Hence $v_k \neq v_i$ for $1 \leq i < k-2$. On the other hand, suppose that $v_k = v_0$. Then $e_{v_0 v_{(k-1)}}$ and $e_{v_0 v_1}$ are the two positive edges intersecting at v_0. In particular, this implies that for each $0 \leq i \leq k-1$, we have found the two positive edges that intersect at vertex v_i. Noting that $n > k-1$, let w be any vertex in γ_h different from $v_0, \ldots, v_{(k-1)}$. It follows from Lemma 5.2 that $e_{v_i w}$ cannot be a positive edge for any $0 \leq i \leq k-1$. This contradicts the assumption that γ_h is a connected set. Hence $v_k \neq v_0$.

Repeating the above step, we identify all the distinct vertices $v_0, \ldots, v_n$ in γ_h such that $e_{v_i v_{(i+1)}}$ is a positive edge for $0 \leq i < n$. All vertices in γ_h can be found this way because γ_h is connected. It only remains to show that $e_{v_n v_0} \subset \gamma_h$. The argument is similar to the one given above. Lemma 5.2 shows that precisely two positive edges intersect at v_n. One of them is $e_{v_{(n-1)} v_n}$. Since $v_0, \ldots, v_n$ are all the vertices in γ_h, the other edge has to be $e_{v_n v_i}$ for some $0 \leq i \leq n-2$. However, $e_{v_i v_n}$ cannot be a positive edge for $1 \leq i < n-1$ since we have already identified $e_{v_{(i-1)} v_i}$ and $e_{v_i v_{(i+1)}}$ as the two positive edges intersecting at v_i. Hence we conclude that $e_{v_n v_0}$ is a positive edge of γ_h. ∎

6. Injectivity on connected components of Γ_h.

The main result of this section is the following lemma.

LEMMA 6.1. *Let γ and γ_h be connected components of Γ and Γ_h, respectively, such that $\gamma \cap \pi(\gamma_h) \neq \emptyset$. Then $\pi : \gamma_h \to \gamma$ is a homeomorphism.*

Surjectivity of $\pi : \gamma_h \to \gamma$ in the above lemma is simple. Continuity of π over the connected set γ_h implies that $\pi(\gamma_h)$ is a connected subset of Γ. Since γ is a connected component of Γ and $\gamma \cap \pi(\gamma_h) \neq \emptyset$, $\pi(\gamma_h) \subseteq \gamma$. We also know that $\pi(\gamma_h)$ is a closed curve because γ_h is a closed curve (Lemma 5.3). Since γ is a Jordan curve, the only closed and connected curve contained in γ is either a point in γ or γ itself. In view of Lemma 4.2, $\pi(\gamma_h)$ is not a point, and hence $\pi(\gamma_h) = \gamma$

The critical step is proving injectivity. For this, we extend the result of Lemma 4.2 in Proposition 6.2 to show that π is injective over any two intersecting positive edges in γ_h (or Γ_h). This result does not suffice for an argument to prove injectivity by considering distinct points in γ_h whose images in γ coincide and then arriving at a contradiction. Instead, we consider a subdivision of γ_h into finitely many connected subsets. For a specific choice of these subsets, we demonstrate using Proposition 6.2 that π is injective over each of these subsets (Proposition 6.3). Then we argue that there can be only one such subset and that it has to equal γ_h itself (Proposition 6.4).

PROPOSITION 6.2. *If e_{ap} and e_{aq} are distinct positive edges in Γ_h, then $\pi : e_{ap} \cup e_{aq} \to \Gamma$ is injective.*

Proof. Let $\alpha_i = \arccos(\hat{N}(\pi(a)) \cdot \hat{U}_{ai})$ for $i = p, q$. By Lemma B.6, we know that $\hat{T}(\pi(a)) \cdot \hat{U}_{ap}$ and $\hat{T}(\pi(a)) \cdot \hat{U}_{aq}$ have opposite (nonzero) signs. Therefore, without loss of generality, assume that $\hat{T}(\pi(a)) \cdot \hat{U}_{ap} < 0$ and $\hat{T}(\pi(a)) \cdot \hat{U}_{aq} > 0$ so that

$$\hat{U}_{ap} = \cos\alpha_p \, \hat{N}(\pi(a)) - \sin\alpha_p \, \hat{T}(\pi(a)), \tag{6.1a}$$

$$\hat{U}_{aq} = \cos\alpha_q \, \hat{N}(\pi(a)) + \sin\alpha_q \, \hat{T}(\pi(a)). \tag{6.1b}$$

We proceed by contradiction. Suppose that x and y are distinct points in $e_{ap} \cup e_{aq}$ such that $\pi(x) = \pi(y)$. By Lemma 4.2, we know that π is injective over e_{ap}, and e_{aq}, respectively. Therefore, x and y cannot both belong to either e_{ap} or e_{aq}. Without loss of generality, assume that $x \in e_{ap} \setminus \{a\}$ and $y \in e_{aq} \setminus \{a\}$. In the following, we identify a point $z \in B(\Gamma, r_n)$ such that $\pi(z)$ equals both $\pi(x)$ and $\pi(a)$. This will contradict Lemma 4.2.

Let $0 < \lambda_x \leq d(a,p)$ and $0 < \lambda_y \leq d(a,q)$ be such that

$$x = a + \lambda_x \hat{U}_{ap} \tag{6.2a}$$

$$\text{and} \quad y = a + \lambda_y \hat{U}_{aq}. \tag{6.2b}$$

Consider the point

$$z = \pi(x) + \xi \hat{N}(\pi(x)), \tag{6.3}$$

$$\text{where} \quad \xi = \frac{\phi(y)\lambda_x \sin\alpha_p + \phi(x)\lambda_y \sin\alpha_q}{\lambda_x \sin\alpha_p + \lambda_y \sin\alpha_q}. \tag{6.4}$$

Since λ_x, λ_y are strictly positive (by definition) and $\sin\alpha_p, \sin\alpha_q$ are strictly positive (Proposition 4.1), we know that $\lambda_x \sin\alpha_p + \lambda_y \sin\alpha_q \neq 0$. Hence z given by (6.3) is well defined. Moreover, from $|\phi(x)| \leq h_K$ and $|\phi(y)| \leq h_K$ (Proposition 3.2), it follows from (6.4) that $|\xi| \leq h_K$. Since $h_K < r_n$ by (3.1a), $z \in B(\Gamma, r_n)$. Therefore from (6.3) and Theorem 2.2, we conclude that $\pi(z) = \pi(x)$.

Next we show that $\pi(z) = \pi(a)$ as well. From Theorem 2.2 and the assumption that $\pi(y) = \pi(x)$, we have

$$x = \pi(x) + \phi(x)\hat{N}(\pi(x)) \tag{6.5a}$$

$$\text{and} \quad y = \pi(y) + \phi(y)\hat{N}(\pi(y)) = \pi(x) + \phi(y)\hat{N}(\pi(x)). \tag{6.5b}$$

Observe from (6.5) that $x \neq y \Rightarrow \phi(x) \neq \phi(y)$. Hence, subtracting (6.5b) from (6.5a), and using (6.2) yields

$$\hat{N}(\pi(x)) = \frac{x - y}{\phi(x) - \phi(y)} = \frac{\lambda_x \hat{U}_{ap} - \lambda_y \hat{U}_{aq}}{\phi(x) - \phi(y)}. \tag{6.6}$$

From (6.2a), (6.3), and (6.5a) we get

$$z = a + \lambda_x \hat{U}_{ap} + (\xi - \phi(x))\hat{N}(\pi(x)). \tag{6.7}$$

Upon using (6.1), (6.4), and (6.6) in (6.7) and simplifying, we get

$$z = a + \underbrace{\frac{\lambda_x \lambda_y \sin(\alpha_p + \alpha_q)}{\lambda_x \sin\alpha_p + \lambda_y \sin\alpha_q}}_{\zeta} \hat{N}(\pi(a)) = \pi(a) + (\phi(a) + \zeta)\,\hat{N}(\pi(a)). \tag{6.8}$$

By Theorem 2.2, (6.8) shows that $\pi(z) = \pi(a)$. Hence we have shown that $\pi(x) = \pi(a)$ (both equal point $\pi(z)$). This contradicts the fact that π is injective on e_{ap}. ∎

To proceed, it is convenient to introduce parameterizations for γ and γ_h. To this end, consider a representation for γ_h as in (5.1), where $\{v_i\}_{i=0}^n$ are all its vertices. From Lemma 5.3 we know that γ_h is a simple, closed curve, so let a parameterization of γ_h be $\alpha : [0, 1) \to \gamma_h$ continuous and one-to-one such that

(i) $\alpha(0) = \alpha(1^-) = v_0$,

(ii) $\alpha^{-1}(v_i) < \alpha^{-1}(v_j)$ if $0 \leq i < j \leq n$.

Clearly $e_{v_i v_{(i+1)}} = \alpha[\alpha^{-1}(v_i), \alpha^{-1}(v_{i+1})]$ for $0 \leq i < n$ and $e_{v_n v_0} = \alpha[\alpha^{-1}(v_n), 1^-)$. Similarly, given that γ is a simple, closed curve, we consider a continuous and one-to-one parameterization $\beta : [0, 1) \to \gamma$ of γ. As discussed at the beginning of this section, the hypotheses in Lemma 6.1 imply that $\pi(\gamma_h) = \gamma$ and in particular that $\pi(v_0) \in \gamma$. Therefore without loss of generality, we assume that $\beta(0) = \beta(1^-) = \pi(v_0)$. For future reference, we note that $\beta^{-1} : \gamma \setminus \pi(v_0) \to (0, 1)$ is injective and continuous as well.

We can now define the connected subsets of γ_h alluded to at the beginning of section 6. Let $P_0 := \{p \in [0, 1) : \pi(\alpha(p)) = \pi(v_0)\}$. Observe that since π is injective over each positive edge in γ_h (Lemma 4.2), each of these edges has at most one point in common with $\alpha(P_0)$. Consequently, P_0 is a collection of finitely many points. Then, noting from the definition of P_0 that $0 \in P_0$, we consider the following ordering for points in P_0:

$$P_0 = \{p_i \,:\, 0 \leq i < m < \infty,\, 0 = p_0 < p_1 < \ldots < p_{m-1} < 1\}. \tag{6.9}$$

Additionally, for convenience we set $p_m = 1$. The connected subsets of γ_h we consider are the sets $\alpha([p_i, p_{i+1}))$ for $0 \leq i < m$.

PROPOSITION 6.3. *For* $0 \leq i < m$, $\pi : \alpha[p_i, p_{i+1}) \to \gamma$ *is a bijection.*

Proof. To prove the proposition, we show that the map $\psi := \beta^{-1} \circ \pi \circ \alpha$ is injective over the interval (p_i, p_{i+1}). To this end, we will need to consider the (positive) edges of γ_h contained in $\alpha[p_i, p_{i+1}]$. Denote the number of such edges by k, set $v_a = \alpha(p_i)$, and define $\{q_j\}_{j=0}^{k+1}$ as $q_j = \alpha^{-1}(v_{a+j})$. Then, by the definition of α, $\{q_j\}_{j=0}^{k+1} \subset [p_i, p_{i+1}]$ and

$$p_i = q_0 < q_1 < \ldots < q_k < q_{k+1} = p_{i+1}. \tag{6.10}$$

Notice that $k \geq 1$ because $k = 0$ would imply that π is not injective on the edge containing the points $\alpha(p_i)$ and $\alpha(p_{i+1})$, contradicting Lemma 4.2.

Consider $0 \leq j \leq k-1$. Proposition 6.2 shows that π is injective over $\alpha[q_j, q_{j+2}]$ and hence ψ is injective over (q_j, q_{j+2}). Since ψ is continuous over (p_i, p_{i+1}), it is continuous over (q_j, q_{j+2}) as well. Consequently, ψ is continuous and strictly monotone over (q_j, q_{j+2}).

From here, we conclude that ψ is continuous and strictly monotone over the interval $(q_0, q_{k+1}) = (p_i, p_{i+1})$. In particular, ψ is injective over (p_i, p_{i+1}). Since β^{-1} is injective over $\gamma \setminus \pi(v_0)$, we get that $\pi \circ \alpha$ is injective over (p_i, p_{i+1}), i.e., that π is injective over $\alpha(p_i, p_{i+1})$. From the definition of P_0, we know that $\pi(\alpha(p_i)) = \pi(v_0)$ and that $\pi(v_0) \notin \pi(\alpha(p_i, p_{i+1}))$. Therefore we conclude that π is in fact injective over $\alpha[p_i, p_{i+1})$.

Finally we show $\pi : \alpha[p_i, p_{i+1}) \to \gamma$ is surjective. Since π is continuous over the connected set $\alpha[p_i, p_{i+1})$, $\pi(\alpha[p_i, p_{i+1}))$ is a connected subset of γ. Since $\pi(\alpha(p_i)) = \pi(\alpha(p_{i+1})) = \pi(v_0)$, $\pi(\alpha[p_i, p_{i+1}))$ equals either $\{\pi(v_0)\}$ or γ. Injectivity of π over $\alpha[p_i, p_{i+1})$ rules out the former possibility. ∎

PROPOSITION 6.4. *Let* P_0 *be as defined in* (6.9). *Then* $P_0 = \{0\}$.

Proof. We prove the proposition by showing that $m > 1$ yields a contradiction. Suppose that $m > 1$. For each $0 \leq i < m$, let $w_i := \alpha(p_i), \gamma_h^i := \alpha[p_i, p_{i+1})$ and define $\Psi_i \colon [0,1) \to \mathbb{R}$ as $\Psi_i := \phi \circ \left(\pi\big|_{\gamma_h^i}\right)^{-1} \circ \beta$. Note that Ψ_i is well defined for each $0 \leq i < m$ because $\pi : \gamma_h^i \to \gamma$ is a bijection from Proposition 6.3. Since it follows from Corollary 4.4 that $\pi^{-1} : \gamma \to \gamma_h^i$ is continuous, we get that Ψ_i is continuous for each $0 \leq i < m$.

For convenience, denote $w_m = v_0 = w_0$. By the definition of P_0, $\pi(w_i) = \pi(v_0)$ for each $0 \leq i \leq m$. From this and Theorem 2.2, we have

$$w_i = \pi(w_i) + \phi(w_i)\, \hat{N}(\pi(w_i)) = \pi(v_0) + \phi(w_i)\, \hat{N}(\pi(v_0)). \tag{6.11}$$

Since $w_i = v_0$ only for $i = 0, m$, (6.11) implies that $\phi(w_i) \neq \phi(v_0)$ for any $1 < i < m$. In particular, since $\phi(w_1) \neq \phi(v_0)$, without loss of generality, assume that $\phi(w_1) > \phi(v_0)$. Then since $\phi(w_m) = \phi(v_0)$, there exists a smallest index k such that (i) $1 \leq k < m$, (ii) $\phi(w_k) \geq \phi(w_1)$, and (iii) $\phi(w_{k+1}) < \phi(w_1)$. For such a choice of k, consider the map $(\Psi_0 - \Psi_k) : [0,1) \to \mathbb{R}$. From $\phi(w_0) = \phi(v_0)$ and $\phi(w_k) \geq \phi(w_1) > \phi(v_0)$, we get

$$(\Psi_0 - \Psi_k)(0) = \phi(w_0) - \phi(w_k) < 0. \tag{6.12}$$

On the other hand, from $\phi(w_{k+1}) < \phi(w_1)$, we get

$$(\Psi_0 - \Psi_k)(1^-) = \phi(w_1) - \phi(w_{k+1}) > 0. \tag{6.13}$$

Equations (6.12) and (6.13) and the continuity of $\Psi_0 - \Psi_k$ on $[0,1)$ imply that there exists $\xi \in (0,1)$ such that $\Psi_0(\xi) = \Psi_k(\xi)$. For this choice of ξ, let $x_0 \in \gamma_h^0$ and $x_k \in \gamma_h^k$ be such that $\pi(x_0) = \pi(x_k) = \beta(\xi)$. That x_0 and x_k exist follows again, from Proposition 6.3. Now notice that $\Psi_0(\xi) = \Psi_k(\xi) \Rightarrow \phi(x_0) = \phi(x_k)$. Therefore from Theorem 2.2, we have

$$x_0 = \pi(x_0) + \phi(x_0)\,\hat{N}(\pi(x_0)) = \pi(x_k) + \phi(x_k)\,\hat{N}(\pi(x_k)) = x_k. \tag{6.14}$$

Equation (6.14) shows that $\gamma_h^0 \cap \gamma_h^k \neq \emptyset$. Since γ_h is a simple curve (Lemma 5.3) and $k \neq 0$, this is a contradiction. ∎

Proof of Lemma 6.1. Propositions 6.3 and 6.4 together show that $\pi : \alpha([0,1)) = \gamma_h \to \gamma$ is a bijection. Since π is continuous on γ_h, it follows from Theorem 4.3 that $\pi : \gamma_h \to \gamma$ is a homeomorphism. ∎

7. Connected components of Γ_h.

The final step in proving part (iv) of Theorem 3.1 is the following lemma.

LEMMA 7.1. *Let γ be a connected component of Γ and let $\gamma_h := \{x \in \Gamma_h : \pi(x) \in \gamma\}$. If $\gamma_h \neq \emptyset$, then γ_h is a simple, closed curve and a connected component of Γ_h.*

To prove the lemma, it suffices to show that γ_h is a connected component of Γ_h, because then Lemma 5.3 would imply that γ_h is a simple, closed curve. To this end, we consider the connected components $\{\gamma_h^i\}_{i=1}^m$ of γ_h. Clearly $m < \infty$. The objective is to demonstrate that γ_h has just one connected component, i.e., that $m = 1$. We do so in simple steps. We first show in Proposition 7.2 that each component γ_h^i is in fact a connected component of Γ_h as well. Next, we order these connected components according to their signed distance from γ (Proposition 7.3). Then, we inspect the relative location of triangles positively cut by each connected component with respect to the rest. This reveals that γ_h has just one connected component.

PROPOSITION 7.2. *For $i \in \{1,\dots,m\}$, each connected component γ_h^i of γ_h is a connected component of Γ_h as well, and consequently*

$$\pi : \gamma_h^i \to \gamma \text{ is a homeomorphism.} \tag{7.1}$$

Proof. Clearly Γ_h has only finitely many connected components, say, $\{\Gamma_h^i\}_{i=1}^k$ for some $k < \infty$. We prove the proposition by demonstrating that for $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,k\}$, $\gamma_h^i \cap \Gamma_h^j \neq \emptyset \Rightarrow \gamma_h^i = \Gamma_h^j$.

Suppose $\gamma_h^i \cap \Gamma_h^j \neq \emptyset$. Then $\pi(\gamma_h^i) \subseteq \pi(\gamma_h) \subseteq \gamma \Rightarrow \pi(\Gamma_h^j) \cap \gamma \neq \emptyset$. Using Lemma 6.1, we get that $\pi : \Gamma_h^j \to \gamma$ is a homeomorphism and, in particular, $\pi(\Gamma_h^j) = \gamma$. By definition of γ_h, we get $\Gamma_h^j \subseteq \gamma_h$. Since $\gamma_h \subseteq \Gamma_h$, Γ_h^j is a connected component of Γ_h and $\Gamma_h^j \subseteq \gamma_h$, we conclude that Γ_h^j is a connected component of γ_h as well. The assumption $\gamma_h^i \cap \Gamma_h^j \neq \emptyset$ implies that Γ_h^j in fact equals γ_h^i. Equation (7.1) follows immediately from Lemma 6.1. ∎

Next, we order the connected components $\{\gamma_h^i\}_{i=1}^m$ of γ_h according to their signed distance from γ. The natural functions to consider for such an ordering are the maps $\Psi_i = \phi \circ \left(\pi\big|_{\gamma_h^i}\right)^{-1}$, $1 \le i \le m$.

PROPOSITION 7.3. *Let $1 \le i, j \le m$. Then the following hold:*

(i) *The function Ψ_i is well defined and continuous, and for $K \in \mathcal{P}_h$ with positive edge $e \subset \gamma_h^i$,*

$$-h_K < \Psi_i(\pi(e)) \le h_K.$$

(ii) *For any $\xi \in \gamma$, $\Psi_i(\xi) = \Psi_j(\xi) \iff i = j$.*

(iii) *If $\Psi_i(\xi) < \Psi_j(\xi)$ for some $\xi \in \gamma$, then $\Psi_i < \Psi_j$ on γ.*

Proof.

(i) The fact that Ψ_i is well defined and continuous is a consequence of (7.1) and the continuity of ϕ. Given positive edge $e \subset \gamma_h^i$ of $K \in \mathcal{P}_h$, part (ii) of Proposition 3.2 shows that $|\Psi_i(\pi(e)| \le h_K$. That $\Psi_i(\pi(e)) > -h_K$ follows from (A.24).

(ii) Let $\xi \in \gamma$ be arbitrary. Following (7.1), let $x_i \in \gamma_h^i$ be such that $\pi(x_i) = \xi$, where $1 \le i \le m$. From $\phi(x_i) = \Psi_i(\xi)$ and Theorem 2.2, we get

$$x_i = \pi(x_i) + \phi(x_i)\,\hat{N}(\pi(x_i)) = \xi + \Psi_i(\xi)\,\hat{N}(\xi). \tag{7.2}$$

Since $\gamma_h^i \cap \gamma_h^j = \emptyset$ for $i \ne j$, $x_i = x_j \iff i = j$. Hence (7.2) implies that $\Psi_i(\xi) = \Psi_j(\xi) \iff i = j$.

(iii) For some $i \ne j$ and $\xi \in \gamma$, assume that $\Psi_i(\xi) < \Psi_j(\xi)$. Suppose there exists $\zeta \in \gamma$ such that $\Psi_i(\zeta) \not< \Psi_j(\zeta)$. Since part (ii) shows $\Psi_i(\zeta) \ne \Psi_j(\zeta)$, we have $\Psi_i(\zeta) > \Psi_j(\zeta)$. Note that $(\Psi_i - \Psi_j)$ is a continuous map on the connected set γ. Therefore, from $(\Psi_i - \Psi_j)(\xi) < 0$, $(\Psi_i - \Psi_j)(\zeta) > 0$ and the intermediate value theorem, we know there exists $\zeta' \in \gamma$ such that $(\Psi_i - \Psi_j)(\zeta') = 0$. This contradicts part (ii). ∎

The above proposition shows that we can find the connected component $i^\sharp$ of γ_h that is closest to γ by simply inspecting the values of $\Psi_j(\xi)$ for $1 \le j \le m$ at any $\xi \in \gamma$. Then, $\Psi_{i^\sharp} < \Psi_j$ on γ for each j different from $i^\sharp$.

As noted previously, each set γ_h^i is a Jordan curve. Hence $\mathbb{R}^2 \backslash \gamma_h^i$ has precisely two connected components, namely, Ω_i^- and Ω_i^+. The purpose of such a decomposition of $\mathbb{R}^2$ is to examine the relative location of the connected components of γ_h and Proposition 7.5 shows how to pick them. To this end, we introduce the curve ω defined as

$$\omega = \{\xi - r_\omega(\xi)\,\hat{N}(\xi) \,:\, \xi \in \gamma\}, \tag{7.3a}$$

$$\text{where} \quad r_\omega = \frac{1}{2}\left(r_n - \Psi_{i^\sharp}\right). \tag{7.3b}$$

We will compare the distances of each connected component γ_h^i of γ_h from γ to establish their relative locations. The curve ω introduced above is useful in these calculations.

PROPOSITION 7.4. *For $\xi \in \gamma$ and $1 \le i \le m$, let $K \in \mathcal{P}_h$ be such that $\left(\pi\big|_{\gamma_h^i}\right)^{-1}(\xi)$ belongs to the positive edge of K. Then*

$$-r_n < \phi(\xi - r_\omega(\xi)\,\hat{N}(\xi)) = -r_\omega(\xi) < \Psi_i(\xi). \tag{7.4}$$

Proof. Following (7.1), we know that there is a unique point $x_i \in \gamma_h^i$ such that $\pi(x_i) = \xi$. Therefore, we can find $K \in \mathcal{P}_h$ such that x_i belongs to the positive edge of K. From part (i) of Proposition 7.3 and (3.1a), we get that $|\Psi_i(\xi)| \leq h_K < r_n$. The definition of r_ω then implies $0 < r_\omega(\xi) < r_n$. Hence Theorem 2.2 shows $\phi(\xi - r_\omega(\xi)\,\hat{N}(\xi)) = -r_\omega(\xi)$. The lower bound in (7.4) follows.

Next, from Proposition 7.3 and (3.1a), we have

$$\Psi_j(\xi) > -h_K > -r_n \quad \text{for } 1 \leq j \leq m. \tag{7.5}$$

Using (7.5) and the definition of $i^\sharp$, we get the upper bound in (7.4):

$$r_\omega(\xi) + \Psi_i(\xi) \geq r_\omega(\xi) + \Psi_{i^\sharp}(\xi) = \frac{1}{2}\left(r_n + \Psi_{i^\sharp}(\xi)\right) > 0 \;\Rightarrow\; -r_\omega(\xi) < \Psi_i(\xi). \qquad \square$$

PROPOSITION 7.5. *For each $1 \leq i \leq m$, $\mathbb{R}^2 \setminus \gamma_h^i$ has precisely two connected components Ω_i^- and Ω_i^+ such that the nonempty set ω is contained in Ω_i^-.*

Proof. First, note that ω is the image of γ under a continuous map. Therefore, the assumption that γ is connected implies that ω is a connected set. Each connected component γ_h^i is a simple, closed curve (Proposition 7.2 and Lemma 5.3). Therefore by the Jordan curve theorem, $\mathbb{R}^2 \setminus \gamma_h^i$ has precisely two connected components. From Proposition 7.4, we know that $-r_\omega < \Psi_i$ on γ. Using this in the definition of ω implies that $\omega \cap \gamma_h^i = \emptyset$. Hence the connected set ω is contained in one of the two connected components of $\mathbb{R}^2 \setminus \gamma_h^i$. The proposition follows from setting Ω_i^- to be the component of $\mathbb{R}^2 \setminus \gamma_h^i$ that contains ω and Ω_i^+ to be the other. $\square$

PROPOSITION 7.6. *For $\xi \in \gamma$ and $i \in \{1, \ldots, m\}$*

$$\emptyset \neq \ell_i^- := \left\{\xi + \lambda\,\hat{N}(\xi) \,:\, -r_n < \lambda < \Psi_i(\xi)\right\} \subset \Omega_i^-. \tag{7.6}$$

Proof. Following (7.1), let $x_i \in \gamma_h^i$ be such that $\pi(x_i) = \xi$. From Theorem 2.2 and $\phi(x_i) = \Psi_i(\xi)$, we have

$$x_i = \xi + \phi(x_i)\hat{N}(\xi) = \xi + \Psi_i(\xi)\hat{N}(\xi). \tag{7.7}$$

Equation (7.7) demonstrates that $x_i \notin \ell_i^-$ and hence that $\ell_i^- \cap \gamma_h^i = \emptyset$. Then, noting that ℓ_i^- is a connected set, either $\ell_i^- \subset \Omega_i^-$ or $\ell_i^- \subset \Omega_i^+$. Therefore, we prove $\ell_i^- \subset \Omega_i^-$ by showing that $\ell_i^- \cap \Omega_i^- \neq \emptyset$. To this end, consider the point $y = \xi - r_\omega(\xi)\hat{N}(\xi)$. While $y \in \omega$ by definition, $-r_n < -r_\omega(\xi) < \Psi_i(\xi)$ from Proposition 7.4 shows that $y \in \ell_i^-$ (and hence $\ell_i^- \neq \emptyset$). Recalling that $\omega \subset \Omega_i^-$ from Proposition 7.5 we get $y \in \ell_i \cap \omega \subset \ell_i \cap \Omega_i^- \Rightarrow \ell_i^- \cap \Omega_i^- \neq \emptyset$. $\square$

PROPOSITION 7.7. *For $\xi \in \gamma$ and $i \in \{1, \ldots, m\}$,*

$$\emptyset \neq \ell_i^+ := \left\{\xi + \lambda\,\hat{N}(\xi) \,:\, \Psi_i(\xi) < \lambda < r_n\right\} \subset \Omega_i^+. \tag{7.8}$$

Proof. The set ℓ_i^+ is nonempty because $\max_\gamma \Psi_i = \max_{\gamma_h^i} \phi < r_n$. By definition, $\ell_i^+ \cap \gamma_h^i = \emptyset$. Since ℓ_i^+ is connected, it is contained either in Ω_i^- or in Ω_i^+. Hence we prove the proposition by demonstrating that $\ell_i^+ \cap \Omega_i^+ \neq \emptyset$.

Following Proposition 7.2, let $x_i \in \gamma_h^i$ be such that $\pi(x_i) = \xi$. Consider first the case in which x_i is not a vertex in γ_h^i. Let e_{ab} be the edge in γ_h^i that contains x_i. Since γ_h^i is a Jordan curve, we know that there exists $\delta > 0$ (possibly depending on x_i) such that $B(x_i, \delta) \cap \gamma_h^i$ is a connected set. Noting that $d(x_i, a), d(x_i, b) > 0$ from $x_i \in \mathbf{ri}(\,e_{ab})$ and that $r_n \pm \Psi_i(\xi) > 0$ from Proposition 7.3 and assumption (3.1a), choose $\varepsilon > 0$ such that

$$\varepsilon < \min\left\{\delta, d(x_i, a), d(x_i, b), r_n \pm \Psi_i(\xi)\right\}. \tag{7.9}$$

In particular, $\varepsilon < \min\{\delta, d(x_i,a), d(x_i,b)\}$ implies that $B(x_i,\varepsilon)\cap\gamma_h^i = B(x_i,\varepsilon)\cap e_{ab}$. Hence, $B(x_i,\varepsilon)\setminus\gamma_h^i$ has precisely two connected components H_- and H_+, defined as $H_\pm = \big(B(x_i,\varepsilon)\setminus\gamma_h^i\big)\cap\Omega_i^\pm$. In particular, H_- is a convex set (being the interior of a half disc).

For the given point $\xi\in\gamma$, let ℓ_i^- be as defined in Proposition 7.6 and set $\zeta_\pm := x_i \pm (\varepsilon/2)\,\hat{N}(\xi)$. From the definition of x_i and $\zeta_\pm$, we get

$$(\zeta_\pm - \xi)\cdot\hat{N}(\xi) = \Psi_i(\xi) \pm \frac{\varepsilon}{2}. \tag{7.10}$$

From (7.10) and $0<\varepsilon<r_n-\Psi_i(\xi)$, we get $\zeta_+\in\ell_i^+\cap B(x_i,\varepsilon)$. Similarly, (7.10) and $0<\varepsilon<r_n+\Psi_i(\xi)$ show that $\zeta_-\in\ell_i^-\cap B(x_i,\varepsilon)$. Using the latter and Proposition 7.6, we get

$$\ell_i^-\subset\Omega_i^- \Rightarrow \ell_i^-\cap B(x_i,\varepsilon)\subset H_- \Rightarrow \zeta_-\in H_-. \tag{7.11}$$

Note that $\zeta_+\neq x_i \Rightarrow \zeta_+\notin\gamma_h^i$. Also, $\zeta_+\in H_-$ yields a contradiction because using $\zeta_-\in H_-$ and the convexity of H_-, we get

$$\zeta_+\in H_- \Rightarrow \frac{1}{2}(\zeta_-+\zeta_+)\in H_- \Rightarrow \gamma_h^i\ni x_i\in H_- \Rightarrow \gamma_h^i\cap\Omega_i^-\neq\emptyset. \tag{7.12}$$

Hence we get the required conclusion that

$$\zeta_+\in H_+ \Rightarrow \ell_i^+\cap H_+\neq\emptyset \Rightarrow \ell_i^+\cap\Omega_i^+\neq\emptyset.$$

The case in which x_i is a vertex is similar. For brevity, we only provide a sketch of the proof and omit details. By Lemma 5.2, precisely two positive edges in γ_h^i intersect at x_i. Let these edges be e_{x_ia} and e_{x_ib}. Choose ε as in (7.9) and define $H_\pm$ as done above. Define $\zeta_\pm$ as above and note that $\zeta_-\in H_-$ as done in (7.11). The main difference compared to the case when x_i is not a vertex is that now, H_- is either a convex or a concave set. If H_- is convex, arguing as in (7.12) shows that $\zeta_+\in H_+$. To show $\zeta_+\in H_+$ when H_- is concave, it is convenient to adopt a coordinate system. The essential step to note from Lemma B.6 is noting that $\hat{T}(\xi)\cdot\hat{U}_{x_ia}$ and $\hat{T}(\xi)\cdot\hat{U}_{x_ib}$ are nonzero and have opposite signs. ∎

Corollary 7.8. *Let $i,j\in\{1,\dots,m\}$. If $\Psi_i(\zeta)<\Psi_j(\zeta)$ for some $\zeta\in\gamma$, then $\gamma_h^j\subset\Omega_i^+$.*

Proof. For an arbitrary point $x\in\gamma_h^j$, let $\xi=\pi(x)$ and define ℓ_i^+ as in (7.8). Since $\Psi_i(\zeta)<\Psi_j(\zeta)$, Proposition 7.3 shows that $\Psi_i(\xi)<\Psi_j(\xi)$. From part (i) of the same proposition and (3.1a), we also know that $\Psi_j<r_n$. Hence we get that $x\in\ell_i^+$. Since $\ell_i^+\subset\Omega_i^+$ (Proposition 7.7), $x\in\ell_i^+\Rightarrow x\in\Omega_i^+$. Since $x\in\gamma_h^j$ was arbitrary, we conclude that $\gamma_h^j\subset\Omega_i^+$. ∎

Proposition 7.9. *Let $i,j\in\{1,\dots,m\}$ and $K=(a,b,c)\in\mathcal{P}_h$ have positive edge $e_{ab}\subset\gamma_h^j$. If $\gamma_h^j\subset\Omega_i^+$, then $\overline{K}\subset\Omega_i^+$.*

Proof. Note that $\gamma_h^j\subset\Omega_i^+$ immediately implies $i\neq j$. Since γ_h^i is a collection of positive edges, the set $K_h^i := \overline{K}\cap\gamma_h^i$ is either empty or a vertex of K or an edge of K. From $i\neq j$, we get

$$e_{ab}\cap\gamma_h^i \subseteq \gamma_h^j\cap\gamma_h^i = \emptyset. \tag{7.13}$$

Therefore, neither a nor b belongs to K_h^i. Hence K_h^i does not contain any edge of K. Since every vertex in γ_h^i has $\phi\geq 0$ but $\phi(c)<0$, $c\notin K_h^i$. Therefore we conclude that $K_h^i=\emptyset$.

Since $\overline{K}$ is a connected set and $K_h^i = \overline{K} \cap \gamma_h^i = \emptyset$, either $\overline{K} \subset \Omega_i^+$ or $\overline{K} \subset \Omega_i^-$. However, $e_{ab} \subset \gamma_h^j \subset \Omega_i^+$ shows that $\overline{K} \cap \Omega_i^+ \neq \emptyset$. Hence $\overline{K} \subset \Omega_i^+$. ∎

PROPOSITION 7.10. *Let $K = (a, b, c) \in \mathcal{T}_h$ and $e_{ab} \subset \gamma_h^i$. Then*

$$\overline{K} \cap \Omega_i^\pm \neq \emptyset \;\Rightarrow\; \overline{K} \setminus \gamma_h^i \subset \Omega_i^\pm. \tag{7.14}$$

Proof. It is convenient to consider the cases $\overline{K} \cap \Omega_i^- \neq \emptyset$ and $\overline{K} \cap \Omega_i^+ \neq \emptyset$ simultaneously. Below we argue by contradiction to demonstrate that $\overline{K} \cap \Omega_i^\pm \neq \emptyset \Rightarrow \left(\overline{K} \setminus \gamma_h^i\right) \cap \Omega_i^\mp = \emptyset$. Then (7.14) follows from recalling that Ω_i^-, Ω_i^+ and γ_h^i are pairwise disjoint and that their union equals $\mathbb{R}^2$.

To this end, let $x \in \overline{K} \cap \Omega_i^\pm$. Since $\gamma_h^i \cap \Omega_i^\pm = \emptyset$, $x \in \left(\overline{K} \setminus \gamma_h^i\right) \cap \Omega_i^\pm$. Suppose there exists $y \in \left(\overline{K} \setminus \gamma_h^i\right) \cap \Omega_i^\mp$. The assumptions $x \in \Omega_i^\pm$ and $y \in \Omega_i^\mp$ imply that line segment joining x and y necessarily intersects γ_h^i. Let point z belong to this intersection. Since $\overline{K} \cap \gamma_h^i$ is a union of one or more edges of K, $\overline{K} \setminus \gamma_h^i$ is a convex set. Therefore $x, y \in \overline{K} \setminus \gamma_h^i \Rightarrow z \in \overline{K} \setminus \gamma_h^i$, which contradicts the fact that $z \in \gamma_h^i$. This proves that $\overline{K} \cap \Omega_i^\pm \neq \emptyset \Rightarrow \left(\overline{K} \setminus \gamma_h^i\right) \cap \Omega_i^\mp = \emptyset$. ∎

Remark 7.11. In the proposition above, $\overline{K} \setminus \gamma_h^i$ can be different from $\overline{K} \setminus e_{ab}$. Of course, if K is positively cut, then neither e_{ac} nor e_{bc} can be a positive edge and the proposition indeed states that $\overline{K} \cap \Omega_i^\pm \neq \emptyset \Rightarrow \overline{K} \setminus e_{ab} \subset \Omega_i^\pm$. However, this need not be the case if K is not positively cut. While γ_h^i being simple (Proposition 7.2) precludes the possibility of all three edges of K being positive edges, it is possible that $K \notin \mathcal{P}_h$ has two positive edges. In such a case, $\overline{K} \setminus \gamma_h^i \subsetneq \overline{K} \setminus \overline{e}_{ab}$.

PROPOSITION 7.12. *Let $K_\pm = (a, b, c_\pm) \in \mathcal{T}_h$ and $1 \leq i \leq m$. If $K_- \in \mathcal{P}_h$ and $e_{ab} \subset \gamma_h^i$, then $\overline{K}_\pm \setminus \gamma_h^i \subset \Omega_i^\pm$.*

Proof. Let $x \in \mathbf{ri}\,(e_{ab})$, $\hat{t} := \hat{T}(\pi(x))$, $\hat{n} := \hat{N}(\pi(x))$, and $\ell := \{\pi(x) + \lambda\,\hat{n} : -r_n < \lambda < r_n\}$. Since Proposition 4.1 shows $|\hat{n} \cdot \hat{U}_{ab}| < 1$ and $h_{K_\pm} < r_n$, we know $\ell \cap K_\pm \neq \emptyset$. Hence pick $y_\pm \in \ell \cap K_\pm$ and note from Proposition B.3 that $\hat{U}_{xy_\pm} = \pm\hat{n}$. Consequently, $y_\pm \in \ell_i^\pm$, whence

$$y_\pm \in \ell_i^\pm \;\Rightarrow\; K_\pm \cap \ell_i^\pm \neq \emptyset \;\Rightarrow\; K_\pm \cap \Omega_i^\pm \neq \emptyset \;\Rightarrow\; \overline{K}_\pm \setminus \gamma_h^i \subset \Omega_i^\pm, \tag{7.15}$$

where we have used Propositions 7.6 and 7.7 for the penultimate and Proposition 7.10 for the last implication. ∎

Proof of Lemma 7.1. We need to show that γ_h has only one connected component, i.e., that $m = 1$. We prove this by supposing that $m > 1$ and arriving at a contradiction. Hence we suppose that γ_h^1 and γ_h^2 are connected components of γ_h. Proposition 7.3 shows that either $\Psi_1 < \Psi_2$ or $\Psi_1 > \Psi_2$ on γ. Without loss of generality, let us assume the former and note using Corollary 7.8 that $\gamma_h^2 \subset \Omega_1^+$.

Consider any positive edge $e_{uv} \subset \gamma_h^2$. By definition, we can find vertex w such that triangle $K = (u, v, w) \in \mathcal{P}_h$ and $\phi(w) < 0$. Since $\gamma_h^2 \subset \Omega_1^+$, Proposition 7.9 in particular shows that $w \in \Omega_1^+$. Below, we demonstrate that $w \in \Omega_1^+ \Rightarrow \phi(w) \geq 0$ to contradict the fact that $\phi(w) < 0$.

From (3.1a), we know that $\xi := \pi(w)$ is well defined. For this choice of ξ, let $\ell_1^\pm$ be as defined in (7.6) and (7.8). From $|\phi(w)| < r_n$ (from Proposition 3.2 and (3.1a)) and $w = \xi + \phi(w)\,\hat{N}(\xi)$, we know $w \in \ell_1^- \cup \{x\} \cup \ell_1^+$. Clearly, $w \neq x$ because x is a point on a positive edge, while w is not. Since $w \in \Omega_1^+$, Proposition 7.6 shows that $w \notin \ell_1^-$. Since $\ell_1^-, \{x\}, \ell_1^+$ are pairwise disjoint, we conclude that $w \in \ell_1^+$. Therefore $\phi(w) > \Psi_1(\xi) = \phi(x)$ and hence

$$\phi(w) = \phi(x) + d(w, x). \tag{7.16}$$

If $\phi(x) \geq 0$, (7.16) together with $x \neq w$ shows that $\phi(w) > 0$, yielding the required contradiction.

The case $\phi(x) < 0$ remains. In the following, we identify a point y such that $\phi(w) > \phi(y) > 0$ to arrive at the required contradiction. To this end, following Proposition 7.2, let $x \in \gamma_h^1$ be such that $\pi(x) = \xi$. Let x belong to a positive edge $e_{ab} \subset \gamma_h^1$. Let $K_- = (a, b, c) \in \mathcal{P}_h$ be the triangle with positive edge e_{ab}; existence of K_- follows from the definition of e_{ab} being a positive edge and uniqueness follows from Lemma 5.1. Since $\phi(a) \geq 0$ and $\phi(x) < 0$, continuity of ϕ on e_{ab} shows that $\phi = 0$ at some point in e_{ab}, i.e., $\exists\, z \in e_{ab} \cap \gamma$. Since γ is immersed in $\mathcal{T}_h$, we can find a sufficiently small $\varepsilon > 0$ such that $B(z, \varepsilon) \subset \mathrm{int}(\omega_h)$, where ω_h is the polygonal domain triangulated by $\mathcal{T}_h$. In particular, the existence of such a ball shows that we can find triangle $K_+ = (a, b, c_+) \in \mathcal{T}_h$ that has edge e_{ab} in common with triangle K_-. From Lemma 5.1, we know that $K_+ \notin \mathcal{P}_h$ and hence that $\phi(c_+) > 0$.

Since $|\hat{N}(\xi) \cdot \hat{U}_{ab}| < 1$ (Proposition 4.1), the line $\ell = \{x + \lambda\,\hat{N}(\xi),\ \lambda \in \mathbb{R}\}$ necessarily intersects either e_{ac_+} or e_{bc_+}. Without loss of generality, let us assume that ℓ intersects e_{ac_+} at point y. Since π is injective on γ_h^1 (Proposition 7.2), $\pi(y) = \xi = \pi(x) \Rightarrow y \notin \gamma_h^1$. Since $y \in \overline{K}_+ \setminus \gamma_h^1$ and Proposition 7.12 shows $\overline{K}_+ \setminus \gamma_h^1 \subset \Omega_1^+$, we know $y \in \Omega_1^+$. Then, repeating the arguments used to show $w \in \ell_1^+$ and (7.16) also demonstrate that $y \in \ell_1^+$ and that

$$\phi(y) = \phi(x) + d(x, y). \tag{7.17}$$

By definition of $\vartheta_{K_-}^{\mathrm{adj}}$ (see Definition 2.4(vi)), the interior angles in K_+ at vertices a and b are greater than or equal to $\vartheta_{K_-}^{\mathrm{adj}}$. Therefore, we have $d(x, y) \geq d(a, x) \sin \vartheta_{K_-}^{\mathrm{adj}}$. Using this and the lower bound for $\phi(x)$ from Corollary A.4 in (7.17), we get

$$\begin{aligned}
\phi(y) &\geq -2C_{K_-}^h\, d(a, x) d(a, b) + d(a, x) \sin \vartheta_{K_-}^{\mathrm{adj}}, \\
&\geq d(a, x) \left(\sin \vartheta_{K_-}^{\mathrm{adj}} - 2C_{K_-}^h\, d(a, b) \right), \\
&\geq d(a, x) \left(\sin \vartheta_{K_-}^{\mathrm{adj}} - 2C_{K_-}^h\, h_{K_-} \right), \\
&> 0 \qquad (\text{using } d(a, x) > 0 \text{ and (3.1d)}).
\end{aligned} \tag{7.18}$$

Now, x, y, and w are collinear points on the line segment $\overline{\ell}_1^+ = \{\xi + \lambda\,\hat{N}(\xi), \Psi_1(\xi) \leq \lambda \leq r_n\}$ with $\lambda = \Psi_1(\xi), \phi(y)$ and $\phi(w)$, respectively. Notice that vertex $w \notin \overline{K}_+$ because $\phi(w) < 0$ while $\phi \geq 0$ at a, b, and c_+. Since $\{\xi + \lambda\,\hat{N}(\xi) : \Psi_1(x) \leq \lambda \leq \phi(y)\} \subset \overline{K}_+$, we conclude that $w \in \{\xi + \lambda\,\hat{N}(\xi) : \phi(y) < \lambda \leq r_n\}$, which in particular shows that $\phi(w) > \phi(y)$. In conjunction with (7.18), we get that $\phi(w) > 0$, yielding the required contradiction.

In this way, we conclude that $m = 1$, i.e., $\gamma_h = \gamma_h^1$. Hence Proposition 7.2 shows that γ_h is a connected component of Γ_h. In turn, Lemma 5.3 implies that γ_h is a simple, closed curve. ∎

Proof of Theorem 3.1. The theorem follows essentially from compiling results we have proved thus far.

(i) See Lemma 5.1.

(ii) For a positive edge $e \subset \Gamma_h$, Lemma 4.5 shows that π is C^1 on $\mathbf{ri}\,(e)$ with the Jacobian bounded away from zero. The inverse function theorem then implies that π is a local C^1-diffeomorphism on $\mathbf{ri}\,(e)$. Since π is injective over $\mathbf{ri}\,(e)$, the assertion follows.

(iii) See Corollary A.4 for lower bound of ϕ and Proposition 3.2 for the upper bound. See Lemma 4.5 for the bounds for the Jacobian.

(iv) With $m \geq 1$, let $\{\gamma^i\}_{i=1}^m$ be the distinct connected components of Γ. For each $i \in \{1, \ldots, m\}$, let $\gamma_h^i := \{x \in \Gamma_h \,:\, \pi(x) \in \gamma^i\}$. By assumption, $\gamma_h^i \neq \emptyset$ for each i. It then follows from Lemma 7.1 that γ_h^i is a simple, closed curve and a connected component of Γ_h, and it follows from Lemma 6.1 that $\pi : \gamma_h^i \to \gamma^i$ is a homeomorphism for each $i \in \{1, \ldots, m\}$.

To show that $\pi : \Gamma_h \to \Gamma$ is a homeomorphism, it is enough to show that it is continuous, one-to-one, and onto (Theorem 4.3). Since $\cup_{i=1}^m \gamma^i = \Gamma$ and $\cup_{i=1}^m \gamma_h^i = \Gamma_h$ by definition, it immediately follows that $\pi : \Gamma_h \to \Gamma$ is continuous and surjective. It only remains to show that $\pi : \Gamma_h \to \Gamma$ is injective. Since we know from Lemma 6.1 that π is injective on each connected component of Γ_h, we only need to consider the possibility that there exist $j, k \in \{1, \ldots, m\}$ such that $j \neq k$ but $\pi(\gamma_h^j) \cap \pi(\gamma_h^k) \neq \emptyset$. Since $\gamma^{j,k} = \pi(\gamma_h^{j,k})$, we have $\gamma^j \cap \gamma^k \neq \emptyset$. Since γ^j and γ^k are connected components of Γ, we in fact get $\gamma^j = \gamma^k$. Then Lemma 7.1 implies that the $\gamma_h^j \cup \gamma_h^k$ is a connected set, which contradicts the fact that γ_h^j and γ_h^k are distinct connected components of Γ_h.

Appendix A. Distance and angle estimates. We prove Proposition 4.1, the essential angle estimate required in section 4 to show injectivity of π over each positive edge and to bound its Jacobian. We begin with a corollary of Proposition 2.3, which is useful when estimating ϕ and $\nabla\phi$ in positively cut triangles while knowing just their values at vertices of the triangle.

Corollary A.1 (of Proposition 2.3). *Let $K \in \mathcal{P}_h$ and $x, y \in \overline{K}$. Then,*

$$\left|\phi(y) - (y - \pi(x)) \cdot \hat{N}(\pi(x))\right| \leq \frac{1}{2} C_K^h d(x,y)^2, \tag{A.1a}$$

$$\text{and} \quad |\nabla\phi(y) - \nabla\phi(x)| \leq C_K^h d(x,y). \tag{A.1b}$$

Proof. Let $L_{xy} \subset \overline{K}$ be the closed line segment joining x and y. We have

$$\max_{L_{xy}} \kappa \circ \pi \leq \max_{\overline{K}} \kappa \circ \pi \leq \max_{\overline{B(K, h_K)} \cap \Gamma} \kappa = M_K \tag{A.2}$$

and $|\phi| \leq h_K$ on L_{xy}. From $\sigma_K C_K^h h_K > 0$ in (3.1c), it follows that $M_K h_K < 1$. Therefore, Proposition 2.3 implies the bound

$$\left|\hat{U}_{xy} \cdot \nabla\nabla\phi(z) \cdot \hat{U}_{xy}\right| \leq \frac{\kappa(\pi(z))}{1 - |\phi(z)|\,\kappa(\pi(z))} \leq \frac{M_K}{1 - M_K h_K} = C_K^h \quad \forall z \in L_{xy}. \tag{A.3}$$

From Taylor's theorem, we have

$$|\phi(y) - \phi(x) - \nabla\phi(x) \cdot (y - x)| \leq \frac{d(x,y)^2}{2} \max_{L_{xy}} \left|\hat{U}_{xy} \cdot \nabla\nabla\phi \cdot \hat{U}_{xy}\right|, \tag{A.4a}$$

$$|\nabla\phi(y) - \nabla\phi(x)| \leq d(x,y) \max_{L_{xy}} \left|\hat{U}_{xy} \cdot \nabla\nabla\phi \cdot \hat{U}_{xy}\right|. \tag{A.4b}$$

Using (A.3) and $x = \pi(x) + \phi(x)\hat{N}(\pi(x))$ (Theorem 2.2) in (A.4) yields (A.1). ◻

Proposition A.2. *Let $K = (a, b, c) \in \mathcal{P}_h$ have positive edge e_{ab}. Then*

$$\hat{N}(\pi(x)) \cdot \hat{U}_{yc} \leq \cos\beta_K \quad \forall x, y \in e_{ab}. \tag{A.5}$$

Proof. Let $\hat{n}_x = \hat{N}(\pi(x))$. From Corollary A.1, we have

$$\phi(i) \le (i - \pi(x)) \cdot \hat{n}_x + \frac{1}{2} C_K^h h_K^2 \quad \text{for } i = a, b \tag{A.6a}$$

$$\text{and} \quad \phi(c) \ge (c - \pi(x)) \cdot \hat{n}_x - \frac{1}{2} C_K^h h_K^2. \tag{A.6b}$$

By definition of η_K in (3.5), we know

$$\phi(i) - \phi(c) \ge \eta_K h_K \quad \text{for } i = a, b. \tag{A.7}$$

Using (A.6) in (A.7), we get

$$(c - i) \cdot \hat{n}_x \le C_K^h h_K^2 - \eta_K h_K \quad \text{for } i = a, b. \tag{A.8}$$

Since $y \in e_{ab}$, y is a convex combination of a and b, (A.8) implies that

$$(c - y) \cdot \hat{n}_x \le C_K^h h_K^2 - \eta_K h_K. \tag{A.9}$$

Dividing (A.9) by $d(c, y)$ and noting that $\rho_K < d(c, y) \le h_K$, we get

$$\hat{U}_{yc} \cdot \hat{n}_x \le C_K^h h_K \frac{h_K}{\rho_K} - \eta_K \frac{h_K}{h_K} = \sigma_K C_K^h h_K - \eta_K = \cos \beta_K, \tag{A.10}$$

which is the required inequality. ∎

PROPOSITION A.3. *Let $K = (a, b, c) \in \mathcal{P}_h$ have positive edge e_{ab} and proximal vertex a. Then*

$$\hat{N}(\pi(a)) \cdot \hat{U}_{ab} \ge -\frac{1}{2} C_K^h h_K. \tag{A.11}$$

Proof. Since a is the proximal vertex of K, $\phi(a) \le \phi(b)$. Then, using Theorem 2.2, we get

$$\phi(b) \ge \phi(a) = (a - \pi(a)) \cdot \hat{N}(\pi(a)). \tag{A.12}$$

From Corollary A.1, we also have

$$\phi(b) \le (b - \pi(a)) \cdot \hat{N}(\pi(a)) + \frac{1}{2} C_K^h d(a, b)^2. \tag{A.13}$$

Comparing (A.12) and (A.13), we get

$$(b - a) \cdot \hat{N}(\pi(a)) \ge -\frac{1}{2} C_K^h d(a, b)^2. \tag{A.14}$$

Dividing (A.14) by $d(a, b)$ and using $d(a, b) \le h_K$ yields

$$\hat{U}_{ab} \cdot \hat{N}(\pi(a)) \ge -\frac{1}{2} C_K^h d(a, b) \ge -\frac{1}{2} C_K^h h_K, \tag{A.15}$$

which is the required inequality. ∎

We can now prove Proposition 4.1.

Proof of Proposition 4.1. We first obtain the lower bound in (4.1) by using the bound for $\hat{N}(\pi(a)) \cdot \hat{U}_{ab}$ derived in Proposition A.3. We have

$$\begin{aligned}
\hat{N}(\pi(x)) \cdot \hat{U}_{ab} &= \hat{N}(\pi(a)) \cdot \hat{U}_{ab} + \Big(\hat{N}(\pi(x)) - \hat{N}(\pi(a))\Big) \cdot \hat{U}_{ab}, \\
&\geq -\frac{1}{2} C_K^h h_K - \Big|\hat{N}(\pi(x)) - \hat{N}(\pi(a))\Big| \qquad \text{(Proposition A.3)} \\
&= -\frac{1}{2} C_K^h h_K - |\nabla\phi(x) - \nabla\phi(a)| \\
&\geq -\frac{1}{2} C_K^h h_K - C_K^h h_K \qquad \text{(Corollary A.1)}\,.
\end{aligned}$$

To derive the upper bound, we make use of the inequality

$$\arccos(\hat{u} \cdot \hat{v}) \leq \arccos(\hat{u} \cdot \hat{w}) + \arccos(\hat{v} \cdot \hat{w}) \tag{A.16}$$

for any three unit vectors $\hat{u}, \hat{v}, \hat{w}$ in $\mathbb{R}^2$ with arccos: $[-1, 1] \to [0, \pi]$. Setting $\hat{u} = \hat{N}(\pi(x))$, $\hat{v} = \hat{U}_{ac}$, and $\hat{w} = \hat{U}_{ab}$ in (A.16), we get

$$\arccos(\hat{N}(\pi(x)) \cdot \hat{U}_{ab}) \geq \arccos(\hat{N}(\pi(x)) \cdot \hat{U}_{ac}) - \arccos(\hat{U}_{ac} \cdot \hat{U}_{ab}). \tag{A.17}$$

From Proposition A.2, we know $\hat{N}(\pi(x)) \cdot \hat{U}_{ac} \leq \cos\beta_K$. Since a is the proximal vertex in K, we have $\hat{U}_{ac} \cdot \hat{U}_{ab} = \cos\vartheta_K$. The upper bound in (4.1) follows.

Finally, to demonstrate that $\left|\hat{N}(\pi(x)) \cdot \hat{U}_{ab}\right| < 1$, it suffices to show that $\frac{3}{2} C_K^h h_K$ and $\cos(\beta_K - \vartheta_K)$ are both smaller than 1. The latter follows from part (iv) of Proposition 3.2. For the former, noting that $\sigma_K \geq \sqrt{3}$ in (3.1c) yields $(3/2) C_K^h h_K \leq \sigma_K C_K^h h_K < \sin\vartheta_K / 2 < 1$. ∎

Part (ii) of Proposition 3.2 implies the lower bound $\phi \geq -h_K$ on the positive edge of $K \in \mathcal{P}_h$. This can be improved using the fact that $\phi \geq 0$ at each vertex in Γ_h. The tighter bound computed below is used in section 7.

Corollary A.4 (of Proposition A.3). *Let $K = (a, b, c) \in \mathcal{P}_h$ have positive edge e_{ab}. Then*

$$\phi(x) \geq -2 C_K^h \min\{d(a, x), d(b, x)\}\, d(a, b) \quad \forall x \in e_{ab}. \tag{A.18}$$

Proof. If a is the proximal vertex of K, then (A.15) of the above proposition shows that

$$\hat{U}_{ab} \cdot \hat{N}(\pi(a)) \geq -\frac{1}{2} C_K^h d(a, b). \tag{A.19}$$

Otherwise, b is the proximal vertex of K and we have

$$\begin{aligned}
\hat{U}_{ab} \cdot \hat{N}(\pi(a)) &= \hat{U}_{ab} \cdot \hat{N}(\pi(b)) + \hat{U}_{ab} \cdot \Big(\hat{N}(\pi(a)) - \hat{N}(\pi(b))\Big), \\
&\geq \hat{U}_{ab} \cdot \hat{N}(\pi(b))) - |\nabla\phi(a) - \nabla\phi(a)|, \\
&\geq \hat{U}_{ab} \cdot \hat{N}(\pi(b)) - C_K^h d(a, b) \qquad \text{(from Corollary A.1)} \\
&\geq -\frac{3}{2} C_K^h d(a, b) \qquad \text{(using (A.15))}\,.
\end{aligned} \tag{A.20}$$

From (A.19) and (A.20), we conclude that

$$\hat{U}_{ab} \cdot \hat{N}(\pi(a)) \geq -\frac{3}{2} C_K^h d(a, b). \tag{A.21}$$

Next, using Corollary A.1, we have

$$\begin{aligned}
\phi(x) &\geq (x-\pi(a))\cdot\hat{N}(\pi(a)) - \frac{1}{2}C_K^h d(a,x)^2 \\
&= \phi(a) + (x-a)\cdot\hat{N}(\pi(a)) - \frac{1}{2}C_K^h d(a,x)^2 \quad \Big(\pi(a) = a - \phi(a)\hat{N}(\pi(a))\Big) \\
&\geq -d(a,x)\hat{U}_{ab}\cdot\hat{N}(\pi(a)) - \frac{1}{2}C_K^h d(a,x)^2 \qquad (\phi(a)\geq 0) \\
&\geq -\frac{3}{2}C_K^h d(a,x)d(a,b) - \frac{1}{2}C_K^h d(a,x)^2 \qquad (\text{using (A.21)}) \\
&\geq -2C_K^h d(a,x)d(a,b) \qquad (\text{using } d(a,x) < d(a,b)). \qquad \text{(A.22)}
\end{aligned}$$

Of course, we can interchange the roles of a and b in the above calculations. The required lower bound for $\phi(x)$ follows. ∎

That the lower bound computed above is better than the trivial one $\phi \geq -h_K$ is easily demonstrated. Noting that $\sigma_K \geq \sqrt{3}$ and $\vartheta_K < 90°$ (assumption (3.1b)) in (3.1c) yields

$$C_K^h h_K < \frac{1}{\sqrt{3}}\sin\frac{\vartheta_K}{2} \leq \frac{1}{\sqrt{6}}. \qquad \text{(A.23)}$$

The estimate in (A.18) then implies

$$\phi \geq -C_K^h h_K^2 > -\frac{h_K}{\sqrt{6}}. \qquad \text{(A.24)}$$

Appendix B. About the set of positive edges. We prove Lemmas 5.1 and 5.2 here. We proceed in simple steps, starting by examining the orientation of positive edges with respect to the local normal and tangent to Γ. From these calculations, we conclude that each edge in Γ_h is a positive edge of just one positively cut triangle (Lemma 5.1). This result in turn helps us show that at least two positive edges intersect at each vertex in Γ_h (Lemma B.5), a useful step in proving Lemma 5.2. In the following, $\text{sgn}:\mathbb{R}\to\{-1,0,1\}$ is the function defined as $\text{sgn}(x) = x/|x|$ if $x\neq 0$ and $\text{sgn}(x)=0$ if $x=0$.

Proposition B.1. *Let $(a,b,c)\in\mathcal{P}_h$ have positive edge e_{ab} and proximal vertex a. Then*

$$\text{sgn}(\hat{T}(\pi(a))\cdot\hat{U}_{ab}) = \text{sgn}(\hat{T}(\pi(a))\cdot\hat{U}_{ac}) \neq 0, \qquad \text{(B.1a)}$$

$$\hat{N}(\pi(a))\cdot\hat{U}_{ac} < \hat{N}(\pi(a))\cdot\hat{U}_{ab}. \qquad \text{(B.1b)}$$

Proof. For convenience, let $\hat{t} = \hat{T}(\pi(a))$ and $\hat{n} = \hat{N}(\pi(a))$. Let $\alpha_b, \alpha_c \in [0°, 360°)$ denote the angles from $\hat{n}$ to $\hat{U}_{ab}$ and $\hat{U}_{ac}$ respectively measured in the clockwise sense so that

$$\hat{U}_{ai} = \cos\alpha_i\,\hat{n} + \sin\alpha_i\,\hat{t} \quad \text{for } i = b, c. \qquad \text{(B.2)}$$

From (B.2) and the assumption that a is the proximal vertex in K, note that

$$\cos\vartheta_K = \hat{U}_{ab}\cdot\hat{U}_{ac} = \cos\alpha_b\cos\alpha_c + \sin\alpha_b\sin\alpha_c = \cos(\alpha_c - \alpha_b). \qquad \text{(B.3)}$$

First we prove (B.1a). Since Proposition 4.1 shows $\hat{t}\cdot\hat{U}_{ab}\neq 0$, without loss of generality assume that $\hat{t}\cdot\hat{U}_{ab} > 0 \Rightarrow \alpha_b \in (0°, 180°)$. The upper bound can be improved by invoking Proposition A.3, (3.1c), and $\sigma_K \geq \sqrt{3}$:

$$\cos\alpha_b = \hat{n}\cdot\hat{U}_{ab} \geq -\frac{1}{2}C_K^h h_K \geq -\sigma_K C_K^h h_K > -\cos\vartheta_K \Rightarrow \alpha_b < 180^\circ - \vartheta_K. \tag{B.4}$$

Suppose then that $\hat{t}\cdot\hat{U}_{ac} \leq 0$, i.e., $\alpha_c \geq 180^\circ$. From Propositions 3.2 and A.2, we have $\alpha_c \leq 360^\circ - \beta_K < 360^\circ - \vartheta_K$. In conjunction with (B.4), this shows $\vartheta_K \leq (\alpha_c - 180^\circ) + \vartheta_K < \alpha_c - \alpha_b < 360^\circ - \vartheta_K$, which clearly contradicts (B.3). Therefore $\hat{t}\cdot\hat{U}_{ab} > 0 \Rightarrow \hat{t}\cdot\hat{U}_{ac} > 0$ as well. The case $\hat{t}\cdot\hat{U}_{ab} < 0$ is argued similarly.

Next we show (B.1b). Following (B.1a), without loss of generality assume that $\hat{t}\cdot\hat{U}_{ab}$ and $\hat{t}\cdot\hat{U}_{ac}$ are both positive. Consequently, $\alpha_b, \alpha_c \in (0^\circ, 180^\circ)$. We proceed by contradiction. Suppose that $\hat{n}\cdot\hat{U}_{ab} \leq \hat{n}\cdot\hat{U}_{ac} \Rightarrow \alpha_c \leq \alpha_b$. Then, noting that $\cos\beta_K < \sigma_K C_K^h h_K$ (from (3.4) and part (iii) of Proposition 3.2), $\cos\alpha_c \leq \cos\beta_K$ (Proposition A.2), and $\cos\alpha_b \geq -\sigma_K C_K^h h_K$ (Proposition A.3, $\sigma_K \geq \sqrt{3}$), we get

$$90^\circ - \arcsin(\sigma_K C_K^h h_K) < \beta_K \leq \alpha_c \leq \alpha_b \leq 90^\circ + \arcsin(\sigma_K C_K^h h_K), \tag{B.5}$$

where $\arcsin\colon [-1,1] \to [-\pi/2, \pi/2]$. Together with (3.1c), this implies that $\alpha_b - \alpha_c < 2\arcsin(\sigma_K C_K^h h_K) < 2\times\vartheta_K/2 = \vartheta_K$, which contradicts (B.3), and hence $\hat{n}\cdot\hat{U}_{ab} > \hat{n}\cdot\hat{U}_{ac}$. Again, the case in which both terms in (B.1a) are negative is handled similarly. ∎

PROPOSITION B.2. *Let $(a,b,c) \in \mathcal{P}_h$ have positive edge e_{ab}. Then*

$$\operatorname{sgn}(\hat{T}(\pi(x))\cdot\hat{U}_{ab}) = \operatorname{sgn}(\hat{U}_{ca}\cdot\hat{U}_{ab}^{\perp}) = \operatorname{sgn}(\hat{U}_{cb}\cdot\hat{U}_{ab}^{\perp}) \quad \forall x \in e_{ab}. \tag{B.6}$$

Proof. Notice first that since

$$d(c,a)\hat{U}_{ca} = d(c,b)\hat{U}_{cb} + d(b,a)\hat{U}_{ba}, \tag{B.7}$$

it follows that $\operatorname{sgn}(\hat{U}_{ca}\cdot\hat{U}_{ab}^{\perp}) = \operatorname{sgn}(\hat{U}_{cb}\cdot\hat{U}_{ab}^{\perp})$ after taking the inner product on both sides with $\hat{U}_{ab}^{\perp}$. Without loss of generality then, assume that the proximal vertex in triangle (a,b,c) is the vertex a. For convenience, let $\alpha_i = \arccos(\hat{N}(\pi(a))\cdot\hat{U}_{ai})$ for $i = b, c$. From Proposition B.1, we know $\operatorname{sgn}(\hat{U}_{ab}\cdot\hat{T}(\pi(a))) = \operatorname{sgn}(\hat{T}(\pi(a))\cdot\hat{U}_{ac}) := \iota$. From the definition of α_b, α_c, and ι, we have

$$\hat{U}_{ai} = \cos\alpha_i\ \hat{n} + \iota\sin\alpha_i\ \hat{t} \quad \text{for } i = b, c, \tag{B.8a}$$

$$\hat{U}_{ab}^{\perp} = \iota\sin\alpha_b\ \hat{n} - \cos\alpha_b\ \hat{t}, \tag{B.8b}$$

where we have again set $\hat{t} = \hat{T}(\pi(a))$ and $\hat{n} = \hat{N}(\pi(a))$. Noting that $0^\circ < \alpha_b < 180^\circ$ from Proposition 4.1 and $\alpha_b < \alpha_c$ from Proposition B.1, we get $0^\circ < \alpha_c - \alpha_b < 180^\circ$. Then, using (B.8), we have the calculation

$$\operatorname{sgn}(\hat{U}_{ca}\cdot\hat{U}_{ab}^{\perp}) = \operatorname{sgn}(\iota\sin(\alpha_c - \alpha_b)) = \iota = \operatorname{sgn}(\hat{t}\cdot\hat{U}_{ab}), \tag{B.9}$$

which proves (B.6) for $x = a$. This in fact implies (B.6) for every $x \in e_{ab}$. For if we suppose otherwise, then by continuity of the mapping $\hat{U}_{ab}\cdot(\hat{T}\circ\pi) : e_{ab} \to \mathbb{R}$, there would exist $y \in e_{ab}$ such that $\hat{U}_{ab}\cdot\hat{T}(\pi(y)) = 0$, contradicting Proposition 4.1. ∎

Proof of Lemma 5.1. Let e_{ab} be a positive edge in Γ_h. By definition, we can find $K = (a,b,c) \in \mathcal{P}_h$ for which e_{ab} is a positive edge. Suppose that there exists $\tilde{K} = (a,b,d) \in \mathcal{P}_h$ different from K that also has positive edge e_{ab}. Then, applying Proposition B.2 to triangles K and $\tilde{K}$, we get

$$\operatorname{sgn}(\hat{U}_{ab}^{\perp}\cdot\hat{U}_{ca}) = \operatorname{sgn}(\hat{U}_{ab}^{\perp}\cdot\hat{U}_{da}), \tag{B.10}$$

because both equal $\text{sgn}(\hat{U}_{ab} \cdot \hat{T}(\pi(a)))$. But (B.10) implies that $K \cap \tilde{K} \neq \emptyset$. This is a contradiction since K and $\tilde{K}$ are nonoverlapping open sets. ∎

PROPOSITION B.3. *Let $K_\pm = (a, b, c_\pm) \in \mathcal{T}_h$ and $K_- \in \mathcal{P}_h$ have positive edge e_{ab}. If $x \in \boldsymbol{ri}(e_{ab})$, then*

$$y \in \{\pi(x) + \lambda\, \hat{N}(\pi(x)) \,:\, \lambda \in \mathbb{R}\} \cap K_\pm \;\Rightarrow\; \hat{U}_{xy} \cdot \hat{N}(\pi(x)) = \pm 1. \tag{B.11}$$

Proof. Denote $\hat{t} := \hat{T}(\pi(x))$ and $\hat{n} := \hat{N}(\pi(x))$. We consider first the case $y \in K_-$. By choice of y, $x \neq y$ and hence $\hat{U}_{xy}$ is well defined. Furthermore, $\hat{U}_{xy}$ is parallel to $\hat{n}$ and hence

$$\hat{U}_{xy} \cdot \hat{n} = \text{sgn}\left(\hat{U}_{xy} \cdot \hat{n}\right) \neq 0. \tag{B.12}$$

From Proposition B.2, we know

$$\text{sgn}\left(\hat{t} \cdot \hat{U}_{ab}\right) = -\text{sgn}\left(\hat{U}_{ab}^{\perp} \cdot \hat{U}_{ac}\right). \tag{B.13}$$

However, $x \in e_{ab}$ and $y \in K_-$ implies

$$\text{sgn}\left(\hat{U}_{ab}^{\perp} \cdot \hat{U}_{xy}\right) = \text{sgn}\left(\hat{U}_{ab}^{\perp} \cdot \hat{U}_{ac}\right). \tag{B.14}$$

Using (B.14) in (B.13) yields

$$\text{sgn}\left(\hat{U}_{ab}^{\perp} \cdot \hat{U}_{xy}\right) = -\text{sgn}(\hat{t} \cdot \hat{U}_{ab}). \tag{B.15}$$

Examining (B.15) above in a local coordinate system leads to the conclusion we seek. To this end, let $\alpha := \arccos\left(\hat{n} \cdot \hat{U}_{ab}\right)$ and note from Proposition 4.1 that $0^\circ < \alpha < 180^\circ$ and $\text{sgn}\big(\hat{t} \cdot \hat{U}_{ab}\big) \neq 0$. We have

$$\hat{U}_{ab} = \cos\alpha\, \hat{n} + \text{sgn}(\hat{t} \cdot \hat{U}_{ab}) \sin\alpha\, \hat{t}, \tag{B.16a}$$

$$\hat{U}_{ab}^{\perp} = \text{sgn}(\hat{t} \cdot \hat{U}_{ab}) \sin\alpha\, \hat{n} - \cos\alpha\, \hat{t}. \tag{B.16b}$$

Evaluating (B.15) using (B.12) and (B.16) yields

$$\text{sgn}\left(\left(\hat{n} \cdot \hat{U}_{xy}\right)\, \left(\hat{t} \cdot \hat{U}_{ab}\right)\, \sin\alpha\right) = -\text{sgn}\left(\left(\hat{t} \cdot \hat{U}_{ab}\right)\, \sin\alpha\right). \tag{B.17}$$

Noting that $\sin\alpha > 0$ and $\hat{t} \cdot \hat{U}_{ab} \neq 0$ in (B.17), we conclude that $\text{sgn}\left(\hat{n} \cdot \hat{U}_{xy}\right) = -1$, i.e., $\hat{U}_{xy} = -\hat{n}$.

Next, consider $y' \in \{\pi(x) + \lambda\, \hat{n} \,:\, \lambda \in \mathbb{R}\} \in K_+$. Observe that since K_- and K_+ are distinct triangles sharing a common edge e_{ab},

$$\text{sgn}\left(\hat{U}_{xy'} \cdot \hat{U}_{ab}^{\perp}\right) = -\text{sgn}\left(\hat{U}_{xy} \cdot \hat{U}_{ab}^{\perp}\right). \tag{B.18}$$

Using $\hat{U}_{xy} = -\hat{n}$ and $\hat{n} \cdot \hat{U}_{ab}^{\perp} \neq 0$ (from (B.16b)) in (B.18) shows $\hat{U}_{xy'} = \hat{n}$, which is the required result. ∎

PROPOSITION B.4. *Let e_{pq} be an edge in $\mathcal{T}_h$ such that $\phi(p) \geq 0$ and $\phi(q) < 0$. Then e_{pq} is an edge of two distinct triangles in $\mathcal{T}_h$.*

Proof. Let ω_h be the domain triangulated by $\mathcal{T}_h$. To prove the lemma, it suffices to find a nonempty open ball centered at any point in e_{pq} and contained in ω_h. To this end, note that since ϕ is continuous on e_{pq} and has opposite signs at vertices p and q, we can find $\xi \in \Gamma \cap e_{pq}$. Since Γ is assumed to be immersed in $\mathcal{T}_h$, we know that $\Gamma \subset \text{int}(\omega_h)$. Therefore, there exists $\varepsilon > 0$ such that $B(\xi, \varepsilon) \subset \text{int}(\omega_h)$, which is the required ball. ∎

The following lemma is the essential step in showing that connected components of Γ_h are closed curves.

LEMMA B.5. *At least two positive edges intersect at each vertex in* Γ_h.

Proof. Let a be any vertex in Γ_h. Since Γ_h is the union of positive edges in $\mathcal{T}_h$, it follows that a is a vertex of at least one positive edge. Suppose that a is a vertex of just one positive edge, say, e_{ab_0}. Then, we can find a triangle $(a, b_0, b_1) \in \mathcal{P}_h$ that has positive edge e_{ab_0}. Since $\phi(a) \geq 0$ and $\phi(b_1) < 0$, applying Proposition B.4 to edge e_{ab_1} shows that there exists $(a, b_1, b_2) \in \mathcal{T}_h$ different from (a, b_0, b_1). Since e_{ab_2} is not a positive edge, we know $\phi(b_2) < 0$. Repeating this step, we find distinct vertices $b_1, b_2, \ldots b_n$ such that $(a, b_i, b_{(i+1)}) \in \mathcal{T}_h$ for $i = 0$ to $n-1$, $\phi(b_i) < 0$ for $i = 1$ to $n-1$ and terminate when b_n coincides with b_0. That n is finite follows from the assumption of finite number of vertices in $\mathcal{T}_h$. In particular, we have shown that (a, b_0, b_1) and (a, b_{n-1}, b_0) are distinct triangles in $\mathcal{T}_h$ that both are positively cut by Γ and have positive edge e_{ab_0}. This contradicts Lemma 5.1. ∎

LEMMA B.6. *If* e_{ap} *and* e_{aq} *are distinct positive edges in* $\mathcal{T}_h$, *then*

$$\text{sgn}(\hat{U}_{ap} \cdot \hat{T}(\pi(a))) = -\text{sgn}(\hat{U}_{aq} \cdot \hat{T}(\pi(a)) \neq 0. \tag{B.19}$$

To prove the lemma, we will use the following corollary of Proposition B.2. Note that unlike Proposition B.1, a need not be the proximal vertex in the result below.

COROLLARY B.7 (of Proposition B.2). *Let* $(a, b, c) \in \mathcal{P}_h$ *have positive edge* e_{ab} *and denote* $\hat{t} = \hat{T}(\pi(a))$ *and* $\hat{n} = \hat{N}(\pi(a))$. *Then*

$$\text{sgn}(\hat{t} \cdot \hat{U}_{ab}) = \text{sgn}(\hat{t} \cdot \hat{U}_{ac}) \Rightarrow \hat{n} \cdot \hat{U}_{ab} > \hat{n} \cdot \hat{U}_{ac}. \tag{B.20}$$

Proof. Let $\text{sgn}(\hat{t} \cdot \hat{U}_{ab}) = \text{sgn}(\hat{t} \cdot \hat{U}_{ac}) = \iota$ and $\alpha_i = \arccos(\hat{n} \cdot \hat{U}_{ai})$ for $i = b, c$. Using

$$\hat{U}_{ca} \cdot \hat{U}_{ab}^{\perp} = -\left(\cos\alpha_c\, \hat{n} + \iota\, \sin\alpha_c\, \hat{t}\right) \cdot \left(\iota \sin\alpha_b\, \hat{n} - \cos\alpha_b\, \hat{t}\right) = \iota \sin(\alpha_c - \alpha_b)$$

and Proposition B.2, we get

$$\iota = \text{sgn}(\hat{t} \cdot \hat{U}_{ab}) = \text{sgn}(\hat{U}_{ca} \cdot \hat{U}_{ab}^{\perp}) = \text{sgn}(\iota \sin(\alpha_c - \alpha_b)) = \iota\ \text{sgn}(\sin(\alpha_c - \alpha_b)). \tag{B.21}$$

Since $\iota \neq 0$ from Proposition 4.1, and $\sin(\alpha_c - \alpha_b) \neq 0$ because edges e_{ab} and e_{ac} in triangle (a, b, c) cannot be parallel, we conclude that $\text{sgn}(\sin(\alpha_c - \alpha_b)) = 1$. Hence $\alpha_c > \alpha_b$. ∎

Proof of Lemma B.6. We proceed by contradiction. Let $\hat{t} = \hat{T}(\pi(a))$ and $\hat{n} = \hat{N}(\pi(a))$. Proposition 4.1 shows that neither term in (B.19) equals zero. Therefore, without loss of generality, suppose that

$$\text{sgn}(\hat{t} \cdot \hat{U}_{ap}) = \text{sgn}(\hat{t} \cdot \hat{U}_{aq}) = 1. \tag{B.22}$$

Since e_{ap} and e_{aq} are distinct edges, (B.22) implies that $\hat{n} \cdot \hat{U}_{ap} \neq \hat{n} \cdot \hat{U}_{aq}$. Therefore, without loss of generality, we assume that

$$\hat{n} \cdot \hat{U}_{ap} > \hat{n} \cdot \hat{U}_{aq}. \tag{B.23}$$

Let $\{p_1, \ldots, p_n\}$ be a clockwise enumeration of all vertices in $\mathcal{T}_h$ such that e_{ap_i} is an edge in $\mathcal{T}_h$ for each $i = 1$ to n and $p_1 = p$. Let $m \leq n$ be such that $q = p_m$. Without loss of generality, we assume that e_{ap_i} is not a positive edge for $i = 2$ to $m-1$. Denote by $\alpha_i \in [0°, 360°)$ the angle between $\hat{n}$ and $\hat{U}_{ap_i}$ measured in the clockwise sense. From (B.22) and (B.23), we get that $0° < \alpha_1 < \alpha_m < 180°$. Using the clockwise ordering of vertices, this implies that

$$0° < \alpha_1 < \alpha_2 < \ldots < \alpha_m < 180°. \tag{B.24}$$

Arguing by contradiction, we now show that $(a, p_1, p_2) \in \mathcal{T}_h$ and is positively cut. Suppose that $(a, p_1, p_2) \notin \mathcal{P}_h$, which allows also for the possibility that $(a, p_1, p_2) \notin \mathcal{T}_h$ when p_1 and p_2 are not joined by an edge. Then since e_{ap_1} is a positive edge, $(a, p_n, p_1) \in \mathcal{T}_h$ and is positively cut. Note that the interior angle at a in (a, p_n, p_1), namely, the angle between edges e_{ap_n} and e_{ap_1} measured in the clockwise sense, has to be smaller than 180°. Therefore, either $\alpha_n < \alpha_1$ or $\alpha_n - \alpha_1 > 180°$. In either case, we have

(B.25)
$$\hat{U}_{p_n a} \cdot \hat{U}^{\perp}_{ap_1} = -(\cos\alpha_n\,\hat{n} + \sin\alpha_n\,\hat{t}) \cdot (\sin\alpha_1\,\hat{n} - \cos\alpha_1\,\hat{t}) = \sin(\alpha_n - \alpha_1) < 0.$$

Using Proposition B.2 in (a, p_1, p_n), (B.22) and (B.25), we get

$$1 = \operatorname{sgn}(\hat{U}_{ap_1} \cdot \hat{t}) = \operatorname{sgn}(\hat{U}_{p_n a} \cdot \hat{U}^{\perp}_{ap_1}) = -1,$$

which is a contradiction. Hence, we conclude that $(a, p_1, p_2) \in \mathcal{T}_h$ and is positively cut.

Triangle (a, p_1, p_2) being positively cut with positive edge e_{ap_1} implies $\phi(p_2) < 0$. Then Proposition B.4 shows that $(a, p_2, p_3) \in \mathcal{T}_h$. If $m \neq 3$, then $\phi(p_3) < 0$ since e_{ap_3} is not a positive edge. Repeating this step, we show that $(a, p_i, p_{(i+1)}) \in \mathcal{T}_h$ for $i = 1$ to $m-1$ and that $\phi(p_i) < 0$ for $i = 2$ to $m-1$. In particular, we get that $(a, p_{m-1}, p_m) \in \mathcal{T}_h$ and is positively cut. This contradicts Corollary B.7 because (B.24) shows that $\operatorname{sgn}(\hat{t} \cdot \hat{U}_{ap_{m-1}}) = \operatorname{sgn}(\hat{t} \cdot \hat{U}_{ap_m})$ and $\hat{n} \cdot \hat{U}_{ap_{m-1}} > \hat{n} \cdot \hat{U}_{ap_m}$.

An identical argument with an anticlockwise ordering of vertices applies to the case when $\hat{t} \cdot \hat{U}_{ap}$ and $\hat{t} \cdot \hat{U}_{aq}$ are both strictly negative. ∎

Lemma 5.2 follows immediately from Lemmas B.5 and B.6.

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